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Hacking Bayes Factors With Optional Stopping

im Fach

Kognitionswissenschaften

vorgelegt von

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Abstract

Null Hypothesis Bayesian Testing (NHBT) as an alternative to Null Hypothesis Significance Testing (NHST) has been gaining more attention over the last decade. A desirable property of a testing procedure is its compatibility with Optional Stopping: Collecting data, looking into it and deciding if a decision for a hypothesis can be made or if more data needs to be collected. NHBT seems to allow Optional Stopping and allows to gather evidence not only for the alternative hypothesis, but also for the null hypothesis. If a chosen prior in NHBT does not reflect the belief in the data there is a null hypothesis bias — referred to as the Catch Up Effect. Also it has been shown that in some cases Optional Stopping can be problematic if the prior does not reflect the beliefs in the data. This thesis investigates in which cases the Catch Up Effect occurs and when Optional Stopping is problematic or prone to hacking Bayes Factors. Firstly, the properties of the Catch Up Effect in relation to Bayes Factors are investigated. Secondly, the Optional Stopping procedure with Bayes Factors as a function of sample size is simulated. Finally, examples relating to both concepts are shown to showcase how the "hacking" of these Bayes Factors could be practically possible.

Zusammenfassung

Bayesianisches Nullhypothesentesten (NHBT) hat als Alternative zum Nullhypothesen-Signifikanztesten (NHST) über das letzte Jahrzent mehr Aufmerksamkeit erlangt. Eine erwünschte Eigenschaft eines Testverfahrens ist seine Kompatibilität mit optionalen Stoppen: Daten sammeln, untersuchen und entscheiden, ob eine Entscheidung für eine Hypothese getroffen werden kann oder ob weitere Daten gesammelt werden müssen. NHBT scheint optionales Stoppen zu ermöglichen und erlaubt, nicht nur Evidenz für die Alternativ-, sondern auch für die Nullhypothese zu sammeln. Falls ein gewählter Prior in NHBT nicht die Überzeugungen in den Daten reflektiert, existiert ein Bias für die Nullhypothese — Catch Up Effect genannt. Außerdem wurde gezeigt, dass in einigen Fällen optionales Stoppen problematisch sein kann, falls der Prior nicht die Überzeugungen in den Daten reflektiert. Diese Arbeit untersucht, in welchen Fällen der Catch Up Effect auftritt und wann optionales Stoppen problematisch oder anfällig für das Hacken von Bayes Faktoren ist. Zuerst werden die Eigenschaften des Catch Up Effect in Relation zu den Bayes Faktoren untersucht. Dann wird optionales Stoppen mit Bayes Faktoren als Funktion in Abhängigkeit zur Stichprobengröße simuliert. Zuletzt werden Beispiele in Relation zu den beiden Konzepten gezeigt, um zu demonstrieren wie Bayes Faktoren in der Praxis "gehackt" werden könnten.

Acknowledgments

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Contents

1 Introduction

Null Hypothesis Bayesian Testing (NHBT) with an Bayesian t-test as an alternative to Null Hypothesis Testing (NHST) with the classical t-test gained more attention over the last decade (Tendeiro & Kiers, 2019, p.774). A desirable property of a testing procedure is its compatibility with Optional Stopping: Collecting data, looking into it and deciding if a decision for a hypothesis can be made or if more data needs to be collected (Rouder, 2014, p.301).
For NHBT two main advantages seem to be allowed: Optional Stopping seems to be possible and evidence can not only be gathered for the alternative, but also for the null hypothesis.
This is not possible with NHST, because although it is guaranteed to correctly decide for the alternative hypothesis H_1 for increasing sample size n, if H_1 is indeed true, this is not the case if H_0 is true. If H_0 is true, with increasing sample size n there will be a point, where an incorrect decision for H_1 will be made (see Rouder et al. (2009), p.226). This is a case of *p-hacking* and will be demonstrated later (see Section 2.2).
Rouder et al. (2009) introduced the Bayesian t-test as a symmetrical test based on Bayes Factors. Decisions with Bayes Factors are made with two decision boundaries — one for the null hypothesis model and one for the alternative hypothesis model. For the Bayesian t-test they quantify the evidence of these hypothesis models in form of two different priors — with a point prior for the null hypothesis. Based on these and the observed data the Bayes Factors of the comparing models are computed.
Erven et al. (2007) introduced the term "catch-up phenomenon", which is referred to as the Catch Up Effect. The Catch Up Effect is a non-monotonic behaviour of Bayes Factors in a null and alternative prior Bayes Factor and exists in Bayesian Inference. This bias leans towards the null hypothesis and was shown by Tendeiro and Kiers (2019) to be existent in the Bayesian t-test.
Rouder (2014) claims that for Bayesian Statistics Optional Stopping is indeed possible, since the prior calibration incorporates the posterior odds. Furthermore, Bayes Factors as a measurement of likelihood of model comparison and hypothesis testing in the Optional Stopping procedure are still valid. Even for misspecified models, "resulting updated beliefs may be interpreted as [...] relative plausibility" (Rouder, 2014, p.306).
The "choice of within-model priors is a delicate matter for Bayes Factors" (Tendeiro & Kiers, 2019, p.780). Bayes Factors are comparison factors between two

specified models (in the Bayesian t-test used for null and alternative models) that consist of chosen specified within-model priors. Therefore it is important for Bayesians to choose this prior in form of a density function based on a justified belief. Tendeiro and Kiers (2019) show that this is not an easy matter for an "objective/default prior". These priors were primarily chosen as "objective" and "default", because they have useful theoretical properties. However, they "lack[...] empirical justification for any specific application" (Tendeiro & Kiers, 2019, p.781).

It was also shown by de Heide and Grünwald (2021) that default priors are prone to violations of calibration "somewhat for fixed sample sizes, but much more strongly under optional stopping" (de Heide & Grünwald, 2021). They provide a distinction between different types of priors and propose a critical view of the use of default priors and optional stopping.

The aim of this thesis is to explore the Catch Up Effect and the Optional Stopping procedure to gain insights in the "hackability" of Bayes Factors with respect to default priors. An analysis of the properties of the Catch Up Effect will show, that different parameter changes in the calculation of Bayes Factors will influence the Bayes Factor accordingly. A simulations of the Optional Stopping procedure will show a high bias towards the null hypothesis — especially for small effects. Application examples additionally show how to combine the properties of the Catch Up Effect and Optional Stopping for a better understanding on how Bayes Factors can be hacked practically.

In this thesis it will be shown that Bayes Factors are prone to the Optional Stopping procedure for default priors and that the misspecification of priors or general model assumptions lead to a high bias in decision making. It will also be shown that Optional Stopping is problematic for unjustified default priors and that the properties of the Catch Up Effect can be used to hack Bayes Factors, such that an incorrect decision is made even though an effect exists — and even for settings that are thus far known as reliable.

Two Optional Stopping procedures for NHBT will be defined in this thesis: one for the asymmetrical case and one for the symmetrical case. The asymmetrical case is analogous to the NHST example of p-hacking and has only one decision boundary towards the null hypothesis. Where it is easy to see, that the Asymmetrical Optional Stopping Procedure is problematic to use for gaining insights in collected data, it is questionable if this is also the case for the Symmetrical Optional Stopping Procedure. A focus of the thesis is therefore laid on the Symmetrical Optional Stopping.

Section 2 explains the methods of the thesis and consists of four subsections: important concepts of Bayesian statistics and Bayes Factors are defined in 2.1. Optional Stopping and it shortcomings for NHST are explained in 2.2. The two different Bayesian prior settings are defined for the calculation of Bayes Factors in 2.3 and details on the Optional Stopping simulation can be found in 2.4. The results of the thesis are reported in Section 3. The results of the Catch Up Analysis of important properties and parameter relations are explained in 3.1. The Optional Stopping simulation results are reported in 3.2 with the asymmetrical and symmetrical procedure for the idealised setting (point vs. normal prior) and the symmetrical procedure for a realistic setting (point vs. Cauchy prior). Furthermore an approximation for the idealised setting is also calculated. Finally both approaches — Optional Stopping simulations and Catch Up Effect — will be combined together to show parameter manipulations on a practical example in 3.3.

2 Methods

The methods of this thesis are separated into four parts: The fundamental concepts of Bayes Factor and NHBT are introduced in Section 2.1.Optional Stopping and the underlying mechanism is explained in Section 2.2 . The assumptions for the mathematical settings are laid out with the respective priors in Section 2.3 . And at last the computational simulations for the respective settings are explained in Section 2.4.

2.1 Bayes Factor

The Bayes Theorem — given the probabilities $P(A)$ and $P(B)$ for events A and B as well as the conditional probability $P(B \mid A)$ — infers the probability of $P(A \mid B)$.

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \tag{1}$$

For Bayesians these probabilities are to be interpreted as beliefs about the events before and after evidence about the events is accounted for. The probability before evidence taken in account is the *prior* probability — the initial belief about the event $P(A)$. The probability after evidence taken into account is the *posterior* belief $P(A \mid B)$ given B is true. Given our initial belief about A the support that B provides for A is isolated by $\frac{P(B|A)}{P(B)}$. Therefore for Bayesians the posterior belief updates the initial belief and can be used for future priors. Researchers use the Bayes Theorem as an application for *Bayesian Inference*. Bayesian Inference is the way of using experimental evidence and (multiple) hypotheses to infer support for or against a hypothesis.
A is often interpreted as data D and B as a model or a hypothesis $\mathcal{M}$. $P(\mathcal{M})$ quantifies the prior probability of the model. The probability of the data given the model $P(D \mid \mathcal{M})$— normalized by the probability of the data $P(D)$ — is the weight of the data given the model. Therefore, given a model or hypothesis $\mathcal{M}$ and our gathered data D one can infer the probability of the model given the data $P(\mathcal{M} \mid D)$ with the Bayes Theorem:

$$P(\mathcal{M} \mid D) = \frac{P(D \mid \mathcal{M}) \cdot P(\mathcal{M})}{P(D)} \tag{2}$$

In experiments it is often useful to compare two hypotheses (e.g., null hypothesis and alternative hypothesis). Therefore it is suited to look at the comparison of two models $\mathcal{M}_0$ and $\mathcal{M}_1$ to compare hypotheses in these specified models. To calculate the likelihood of a certain model $P(\mathcal{M}_i \mid D)$ the Bayes Formula is directly applied to $P(\mathcal{M}_i \mid D)$ for $i = 0, 1$:

$$P(\mathcal{M}_i \mid D) = \frac{P(\mathcal{M}_i) \cdot P(D \mid \mathcal{M}_i)}{P(\mathcal{M}_0) \cdot P(D \mid \mathcal{M}_0) + P(\mathcal{M}_1) \cdot P(D \mid \mathcal{M}_1)} \tag{3}$$

Most of the time researchers are interested in the weight the data provides towards a certain model by looking into the change of the *prior odds* to the *posterior odds* by the observed data. The prior odds reflect our beliefs how likely the models are in comparison and are denoted by $\frac{P(\mathcal{M}_1)}{P(\mathcal{M}_0)}$. The posterior odds also reflect our beliefs in the comparison of the likelihood, but after the consideration of the given data D. They are denoted by $\frac{P(\mathcal{M}_1|D)}{P(\mathcal{M}_0|D)}$.
Now for the two given models $\mathcal{M}_0, \mathcal{M}_1$ how much is the weight comparing these models in regards to the observed data for each model? This weight is called the Bayes Factor (BF) and is denoted by $\frac{P(D|\mathcal{M}_1)}{P(D|\mathcal{M}_0)}$. In experiments Bayes Factors are commonly used as a form of hypothesis test to conclude evidence for or against a certain hypothesis.

$$\underbrace{\frac{P(\mathcal{M}_1 \mid D)}{P(\mathcal{M}_0 \mid D)}}_{\text{posterior odds}} = \underbrace{\frac{P(\mathcal{M}_1)}{P(\mathcal{M}_0)}}_{\text{prior odds}} \times \underbrace{\frac{P(D \mid \mathcal{M}_1)}{P(D \mid \mathcal{M}_0)}}_{\text{Bayes Factor } BF_{10}} \tag{4}$$

The Bayes Factor decision threshold BF_{crit} is chosen symmetrical for both hypothesis models. If the calculated Bayes Factor BF_{10} is higher than BF_{crit} one decides for model $\mathcal{M}_1$. If BF_{10} is lower than $1/BF_{\text{crit}}$, one decides for model $\mathcal{M}_0$. Rouder et al. (2009) provide typical critical Bayes Factor decision thresholds: $BF_{\text{crit}} = 3$ and $BF_{\text{crit}} = 10$. The decision on the meaning of these odds-ratio is left to the researcher themselves (see Rouder et al. (2009), p. 231).
A deeper look into the Bayes Factor reveals, that Bayes Factors are sensitive to the choice of the priors set for each model (*within model prior*).
Beliefs about hypothesis are specified as a model by choosing a suitable priors. The advantage of priors is, that every belief about a hypothesis can be modeled with a prior - specified as a density function.

$$P(D \mid \mathcal{M}_i) = \int_{\Theta_i} \underbrace{P(D \mid \theta_i, \mathcal{M}_i)}_{\text{observed data}} \cdot \underbrace{P(\theta_i \mid \mathcal{M}_i)}_{\text{within model prior}} \, d\theta_i \tag{5}$$

For a more detailed introduction to Bayesian statistics see for example Kruschke (2014).

2.2 Optional Stopping

In a common experimental setup researchers have a fixed sample size n and investigate the collected data in this sample.

In Optional Stopping the sample size n is not fixed. Instead, for each collected data point researchers first look at the data and decide, if there is enough evidence to make a decision towards a given hypothesis. If there is not enough evidence yet to make a decision, one continues to sample and repeats this procedure. If there is enough evidence, then the decision for a given hypothesis is made, e.g. to reject the null hypothesis and for the alternative hypothesis.

This practice is not possible for Frequentists, because the Null Hypothesis Significance Test (NHST) is asymmetrical and would lead at some point to a statistically significant p-value.

One can show this with a simple simulation: Suppose data $Y_j, j \in \{1...n\}$ is normally distributed with $Y_j \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$. For the null hypothesis H_0 it is assumed that $\mu = 0$, for the alternative hypothesis H_1 it is assumed that $\mu \neq 0$. A two-sided *t-test* is conducted with the significance level $\alpha = 0.05$. If $p \geq 0.05$ at sample size n, no decision is made and the sample size is increased by one.

One would suppose, if Optional Stopping would work for NHST — then even for an asymmetric test — there would be simply no decision to be made or in 5% of all Optional Stopping cases a wrong decision (Error Type I) would be made. However as one can see in Figure 1 at some sample size n there is a sample size where $p < 0.05$ and therefore decides for H_1 — even though the true effect is at $\mu = 0$ and one should have decided for H_0.

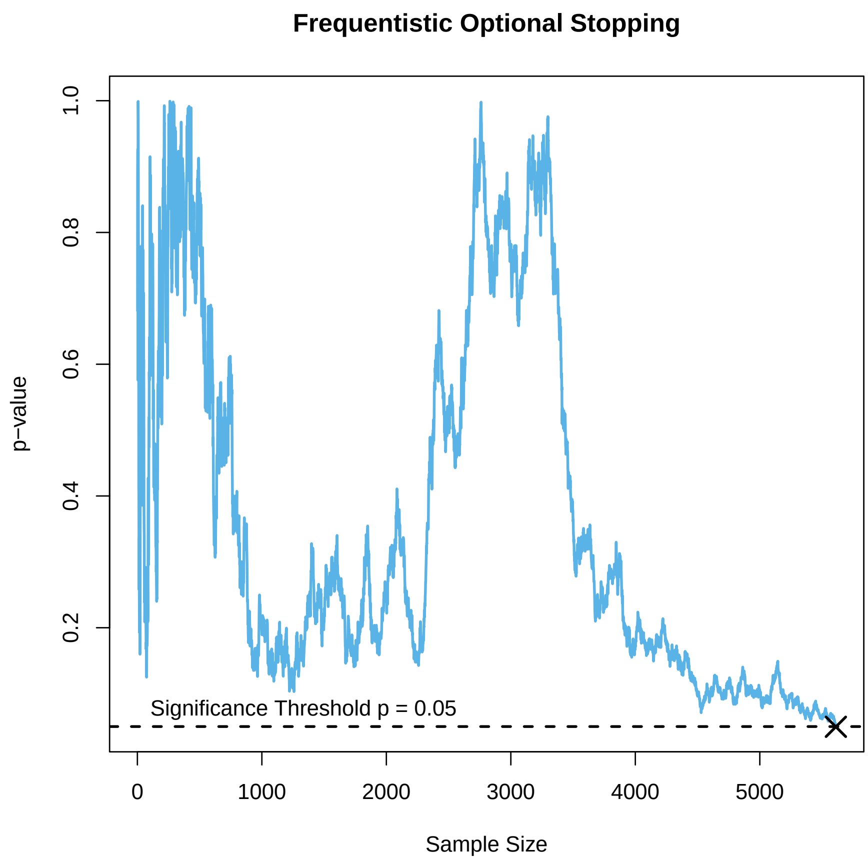


Figure 1: Random Walk for frequentistic asymmetrical significance testing with Optional Stopping from a normal distribution $\mathcal{N}(\mu = 0, \sigma^2 = 1)$.

The inherent problem in the Frequentist Optional Stopping procedure is one of asymmetrical testing, which allows to get as many tries as one needs to incorrectly decide for the alternative hypothesis. This problem can be mapped to Bayesian Optional Stopping with only one decision threshold for H_0. Although, there are decisions for H_0 being made here, this is the same kind of problem.
If one takes a look at the Asymmetrical Optional Stopping Procedure (*Algorithm* 1), one can see that — unless a sufficient sample size is taken as a maximum to stop the Optional Stopping Procedure, one can only decide for H_0 but never for H_1 as stopping without making a decision means, there is just not enough evidence towards H_0. However an indecisive decision can be made —

$H_{1 \text{ or indecisive}}$ — when the sample size maximum is reached without making a decision for H_0.

```
data = collect new data point
sampleSize = length(data)
while BF < BF_crit AND sampleSize < sampleSize_max do
    data = data ++ collect new data point
    BF = calculate new BF based on data
    sampleSize = sampleSize + 1
end
if BF > BF_crit then
    Decide for H_0
end
else
    Decide for H_1 or indecisive
end
```

Algorithm 1: Asymmetrical Optional Stopping Procedure

The commonly used Optional Stopping procedure is the Symmetrical Optional Stopping Procedure (*Algorithm* 2). The decision thresholds are mirrored - if BF_{crit} is the chosen Bayes Factor decision threshold for H_0, then $\frac{1}{BF_{\text{crit}}}$ is the decision threshold for H_1. An example for a random walk can be seen in Figure 2.

```
data = collect new data point
while BF < BF_crit and BF > 1/BF_crit do
    data = data ++ collect new data point
    BF = calculate new BF based on data
end
if BF > BF_crit then
    Decide for H_0
end
if BF < 1/BF_crit then
    Decide for H_1
end
```

Algorithm 2: Symmetrical Optional Stopping Procedure

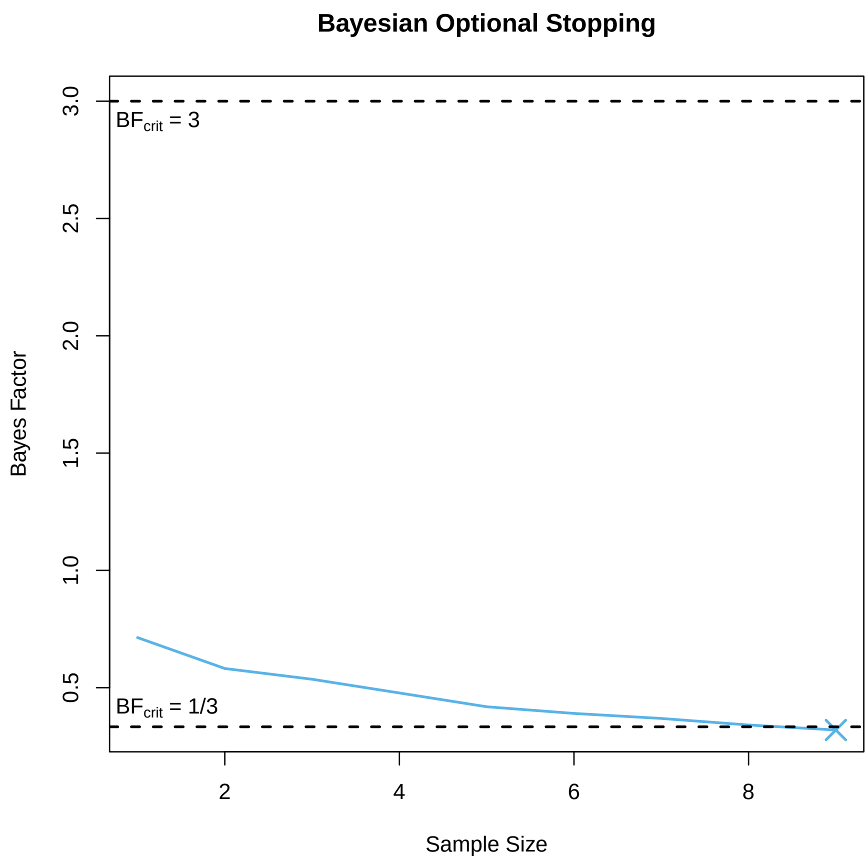


Figure 2: Random Walk for the Symmetrical Optional Stopping Procedure. The Bayes Factors are calculated based on sampling from a normal distribution with $\mathcal{N}(\mu = 0, \sigma^2 = 1)$. Bayes Factors were computed taking the idealised setting as basis (see section 2.3.1).

2.3 Assumptions

In following the idealised and the realistic setting are introduced. The main focus is laid on the idealised setting and analysis for the Catch Up Effect and the Optional Stopping simulation. The realistic setting serves as an extension to the idealised setting and to the observations of the Optional Stopping simulation.

2.3.1 Idealised Setting

Based on the paper of Tendeiro and Kiers (Tendeiro & Kiers, 2019) a specified point prior for the H_0 and a normal prior for the alternative hypothesis H_1 is introduced.
Suppose data is normally distributed with $Y_j \sim \mathcal{N}(\mu, \sigma^2)$ for $j = 1, ..., n$ with known variance σ^2. The two hypotheses are modeled with:

$$
\begin{aligned}
&M_0 : \mu = 0 \text{ (null hypothesis } H_0) \\
&M_1 : \mu \sim \mathcal{N}(0, \sigma_1^2) \text{ (alternative hypothesis } H_1)
\end{aligned}
$$

A visualisation of both models can be seen in Figure 3. Also see Tendeiro and Kiers (2019) for more details on the specification of these models.

Definition 1 (BF10 Formula).
Tendeiro and Kiers (2019) used this setting to calculate the Bayes Factor formula BF_{10} assuming known variance σ^2:

$$BF_{10} = \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \tag{6}$$

Directly from this, BF_{01} can be derived:

$$BF_{01} = \frac{\sqrt{\sigma^2 + n\sigma_1^2}}{\sigma \cdot \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2+n\sigma_1^2)}\right]} \tag{7}$$

The non-monotonic behaviour of the BF_{10} formula with a minimum as bias towards the model M_0 is referred here as Catch Up Effect. This was first introduced by Erven et al. (2007) as "catch-up phenomenon". This Catch Up Effect influences decision making for H_0 under certain circumstances. BF_{10} will be interpreted as a function depending on sample size n to further investigate its properties with respect to Optional Stopping. Figure 4 shows the BF_{10} function and the corresponding Catch Up Effect for different means.
BF_{10} is a continuous function and allows $n \in \mathbb{R}_+$ as inputs. However, data points n are of discrete nature. Therefore in applications it can be useful as a discrete function $BF_{10} : \mathbb{N}_{>0} \to \mathbb{R}_+$.

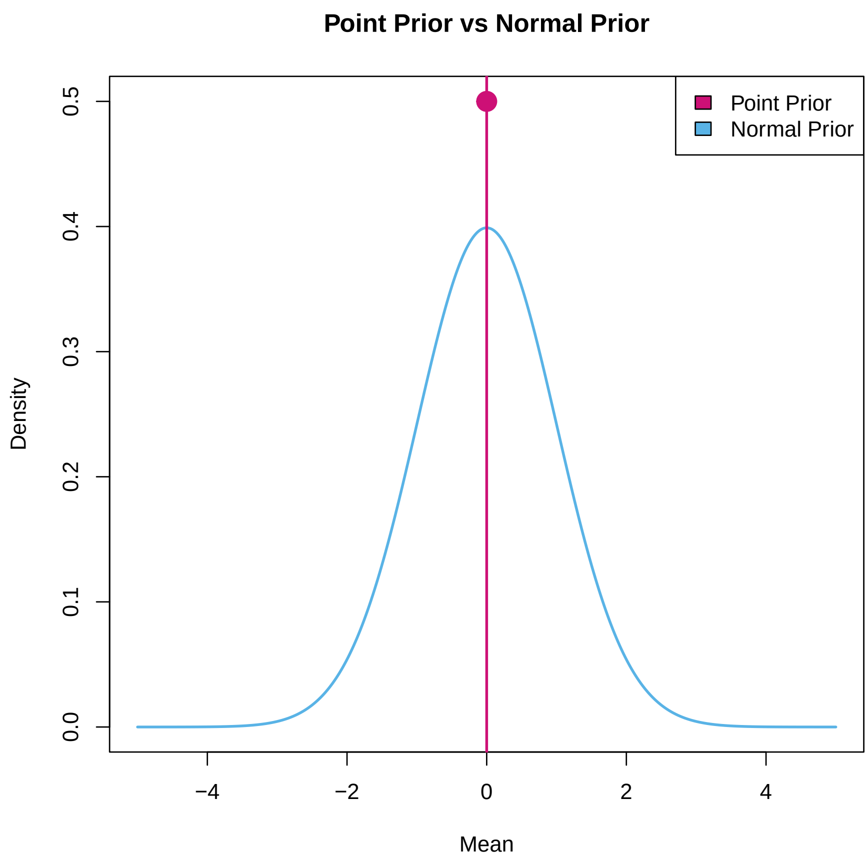


Figure 3: Comparing the null hypothesis (point prior) with the alternative hypothesis (normal prior). The density of the point prior is all at $\mu = 0$ whereas density of the alternative prior is normal distributed.

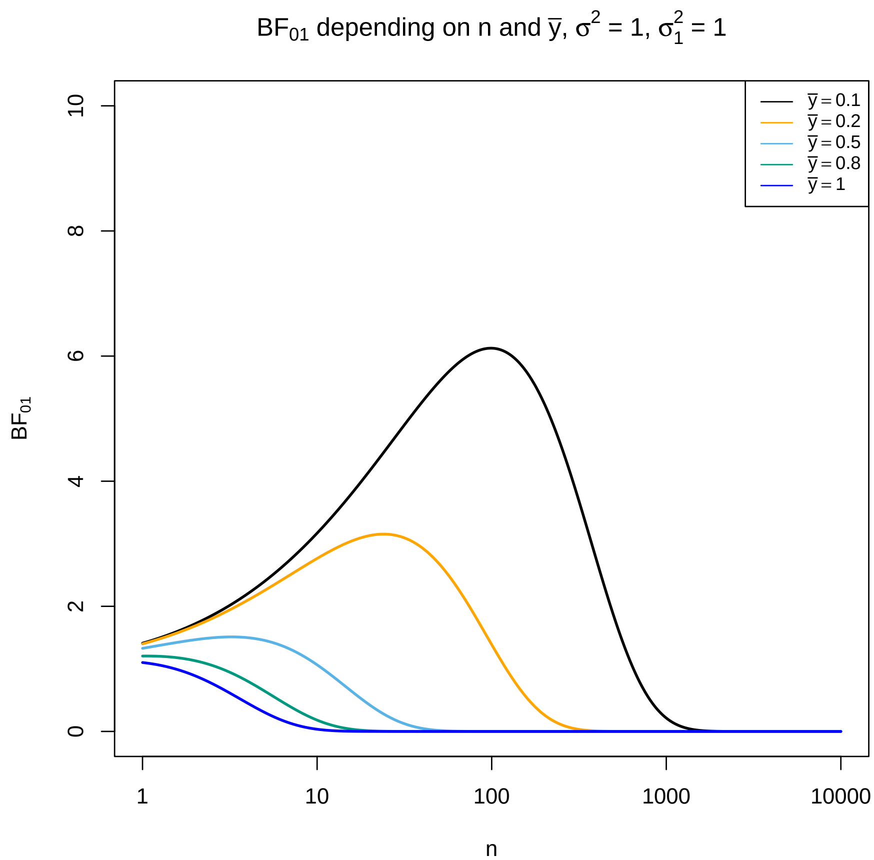


Figure 4: Bayes Factor for different $\bar{y}$ on a log-scaled x-axis with n. Note that BF_{01} is provided instead of BF_{10} here.

Given a critical Bayes Factor threshold BF_{crit} one can change the perspective to look at decision making by comparing the sample mean $\bar{y}$ to a critical mean threshold $\bar{y}_{\text{crit}}$. This will be useful for the proposed approximation in Section 3.2.2.

Definition 2.
Tendeiro and Kiers (2019) derived the inverse Bayes Factor function $\bar{y}_{crit} = BF_{10}^{-1}(n, \sigma, \sigma_1, BF_{crit})$ *with* $BF_{crit} = 1$ *and* $\sigma^2 = 1$. *Generalizing this formula with respect to* BF_{crit} *yields:*

$$\bar{y}_{crit} = \pm \frac{\sqrt{2(1+n\sigma_1^2)}}{n\sigma_1} \left(\ln \left(BF_{crit} \sqrt{1+n\sigma_1^2} \right) \right)^{1/2}$$

Also see appendix A.1 for more details on the derivation.

Also Newtons Method (see Lemma 4) could be useful to iteratively compute $n_{\text{start/end}}$ given a decision threshold $BF_{\text{crit}} = BF_{10(n)}$. This is done to look at intersection points in which it may be more likely for a certain specification of a start sample size n_{start} or maximum sample size n_{end} in Optional Stopping to maximize probability for one of the hypotheses.
The aim of using this idealised setting is the focus on two investigations: Firstly, to investigate the properties of the Bayes Factor function BF_{10} with respect to the Catch Up Effect and to look into meaningful relations between Bayes Factors and other parameters. Secondly, to investigate the decision making for either hypotheses (H_0 or H_1) with both Optional Stopping procedures.

2.3.2 Realistic Setting

Normally the variance is not known. Therefore the idealised setting is "ideal", such that the true variance — including introduced noise — is known. Note, the main focus is on the idealised setting. The realistic setting is only used as an extension to the Optional Stopping simulation. For the realistic setting it is therefore necessary to assume an unknown variance. As an alternative prior instead of a "naive" normal distribution the default prior in form of a Cauchy distribution is commonly used as a weakly informative prior. Therefore, suppose data is still normally distributed with $Y_j \sim \mathcal{N}(\mu, \sigma)$ with an unknown variance σ^2. The Cauchy prior is modeled with the known scale r. Then the two hypotheses are modeled with:

$$M_0 : \mu = 0 \textit{ (null hypothesis } H_0\textit{)}$$
$$M_1 : \mu \sim \text{Cauchy}(0, r) \textit{ (alternative hypothesis } H_1\textit{)}$$

In Figure 5 is a visualisation for the two models compared. Also see Rouder et al. (2009) for more details on the specification of these models.

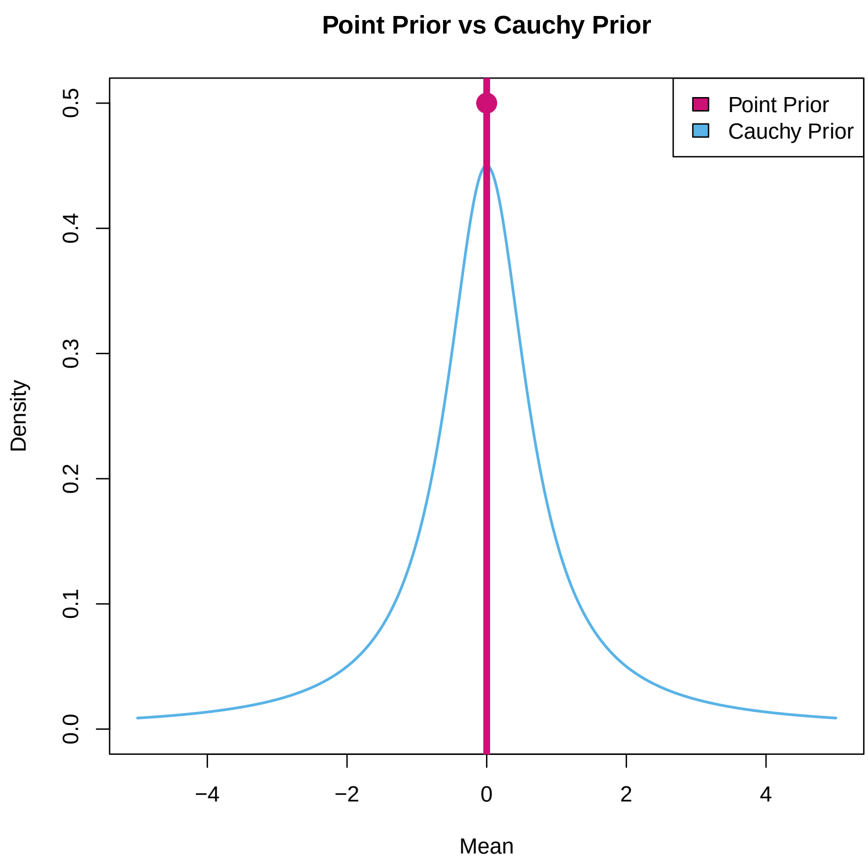


Figure 5: Comparing the null hypothesis (point prior) with the alternative hypothesis (cauchy prior). The cauchy prior has "heavier" tails and allows more extreme outliers than the normal prior.

2.4 Computational Simulation

Simulations are done with R (R Core Team, 2024) and self-implemented functions for the idealised setting considering the algorithms for the asymmetrical (*Algorithm* 1) and the symmetrical Optional Stopping Procedure (*Algorithm* 2). The code corresponding to this thesis is openly available at https://osf.io/a3tcg.

2.4.1 Idealised Simulation

For the Optional Stopping simulation the Idealised Setting is used.
As data is distributed normally, data points are drawn randomly from a normal distribution with the implemented *rnorm* function. Assuming a known true variance of $\sigma^2 = 1$, an alternative prior variance $\sigma_1^2 = 1$ and $\mu = [0, 1]$ with step sizes of $\delta = 0.01$ in between, there are 101 combinations — each one repeated 20000 times. The critical Bayes Factor threshold for H_1 is $BF_{\text{crit}} = 3$. Data gathering is started with one data point and is steadily incremented by one data point. After each data point gathered, the current Bayes Factor is calculated and compared to the critical threshold(s). BF_{10} with the Bayes Factor decision threshold for $H_0 : \frac{1}{BF_{\text{crit}}} = \frac{1}{3}$ and $H_1 : BF_{\text{crit}} = 3$
The asymmetrical Optional Stopping simulation decides via this procedure with $BF_{\text{crit}} = \frac{1}{3}$ for H_0 and after a stopping count of $n = 250$ a decision for $H_{1 \text{ or indecisive}}$ is made.

2.4.2 Realistic Simulation

The realistic simulation is based on the Realistic Setting. The simulation was exactly performed as in Idealised Simulation, but it is now asssumed that the mean in the alternative prior is Cauchy-distributed and therefore a Cauchy prior is chosen: 20000 repetitions and $\mu = [0, 1]$ with step sizes $\delta = 0.01$ in between. Additionally three different scales $r = \{\frac{0.5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\}$ and three different Bayes Factor decision thresholds $BF_{\text{crit}} = \{3, 6, 10\}$ are chosen. Therefore there are in total 909 different combinations with each 20000 repetitions, for which the Symmetrical Optional Stopping Procedure is applied. A simulation for the Asymmetrical Optional Stopping Procedure is discarded as the results of Normal Prior With Known Variance (Idealised Setting) show, that the symmetrical case is more interesting. For the calculation of the Bayes Factors the R package *BayesFactor* (Morey et al., 2015) is used.

3 Results

Results are separated into three parts: Firstly, the Catch Up Effect properties are analysed and parameter relations are derived in Section 3.1. Secondly, the Optional Stopping simulations are reported in Section 3.2. Thirdly, an application example shows how to maximize either the null hypothesis or the alternative hypothesis given Optional Stopping and the parameter relations in Section 3.3.

3.1 Catch Up Effect Analysis

In the Catch Up Effect analysis, properties and relations will be claimed in Propositions with visualisations and later be proven in Proofs.

3.1.1 Propositions

The properties of the Bayes Factor function BF_{10} and especially the Catch Up Effect were analysed and are summarized in the following propositions.

Proposition 1 (Minimum of BF_{10}).
For $n \in (0, \infty), \sigma > 0, \sigma_1 > 0, \bar{y} \neq 0$ the function $BF_{10}(n, \bar{y}, \sigma, \sigma_1)$ (see Definition 1) always has a minimum argument at $\arg\min_n BF_{10}(n, \bar{y}, \sigma, \sigma_1) = n_*$ with

$$n_* = \sigma^2 \left(\frac{-2\bar{y}^2 + \sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2 \bar{y}^2} \right) \tag{8}$$

and a minimum $\min_n BF_{10}(n, \bar{y}, \sigma, \sigma_1) = BF_{10}(n_*, \bar{y}, \sigma, \sigma_1)$ with:

$$\begin{aligned} BF_{10}(n_*) = & \frac{\bar{y}}{\sqrt{\frac{1}{2}\left(\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}\right)}} \\ & \cdot \exp\left[\frac{-\bar{y}^2 + \frac{1}{2}\sigma_1^2 + \sqrt{\bar{y}^4 + \frac{1}{4}\sigma_1^4} + \frac{2\bar{y}^4 - \bar{y}^2\sqrt{4\bar{y}^4 + \sigma_1^4}}{\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}} \right] \end{aligned} \tag{9}$$

As reported in Idealised Setting the Catch Up Effect with its minimum exists as Proposition 1 shows. For a visualisation revisit Figure 4.
The $\arg\min_n \mathrm{BF}_{10}(n, \bar{y}, \sigma, \sigma_1) = n_*$ is not only the best fit for the continuos function BF_{10}, but is also the best fit for for the discrete space, by defining

$$n_{*,\text{discrete}} = \arg \min_{n \in \{\lfloor n_* \rfloor, \lceil n_* \rceil\}} \mathrm{BF}_{10}(n, \bar{y}, \sigma, \sigma_1) \tag{10}$$

This is also the case for other n, that need to be transformed from the continuous to the discrete space.

Corollary 1 (Influence of true standard deviation σ on the minimum of BF_{10}).
For the function BF_{10} the point of the minimum $\arg\min_n BF_{10}$ scales quadratically with the true standard deviation σ. The value of the minimum $\min_n BF_{10}$ is independent of the true standard deviation σ.

The influence of the standard deviation of the normally distributed data does not change the value of the minimum for BF_{10} — instead it scales the location of the minimum quadratically (see Figure 6). This can be directly inferred from its role as the true standard deviation of the normal distributed data. If there is more "noise" in the data there is a higher sample size needed to reach the same likelihood towards a decision.

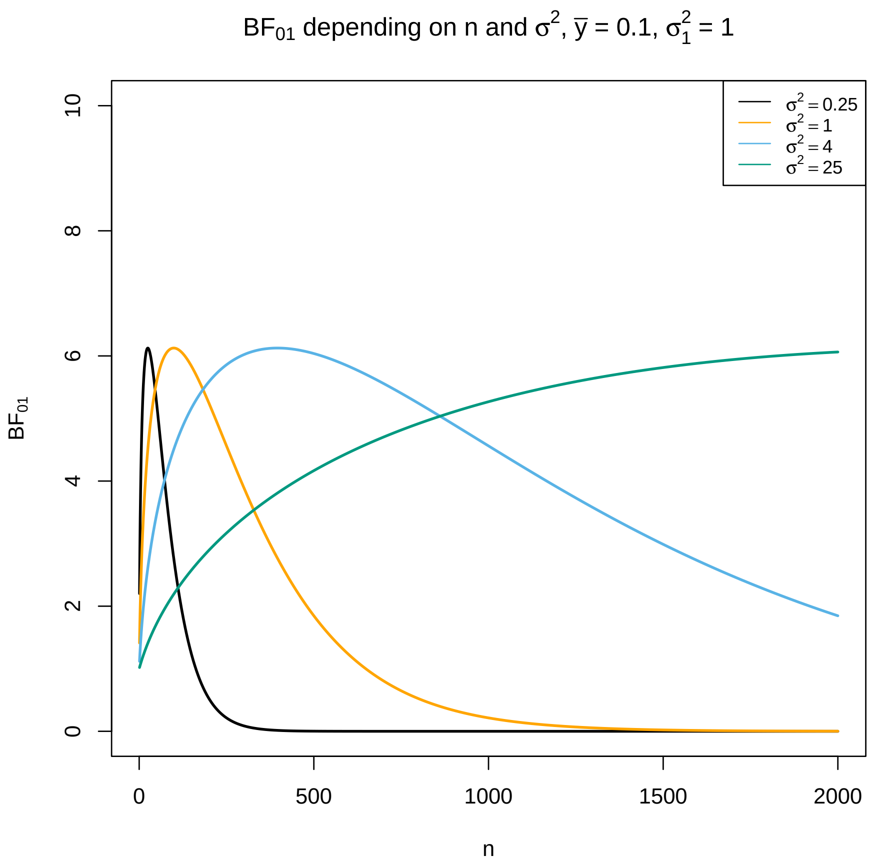


Figure 6: Comparing different variances across fixed parameters.

Corollary 2 (Influence of mean $\bar{y}$ on BF_{10}).
For $n \in (0, \infty), \sigma > 0, \sigma_1 > 0$ the function BF_{10} increases strictly monotonically with respect to $|\bar{y}|$ for all means $\bar{y} \neq 0$.

The Bayes Factor function BF_{10} increases with the amount of the mean — this is not surprising as it is a desirable property following from the central limit theorem — for increasing sample size n to have an increasing probability to decide for the alternative hypothesis H_1 (if an effect exists). Also look at Figure 7.

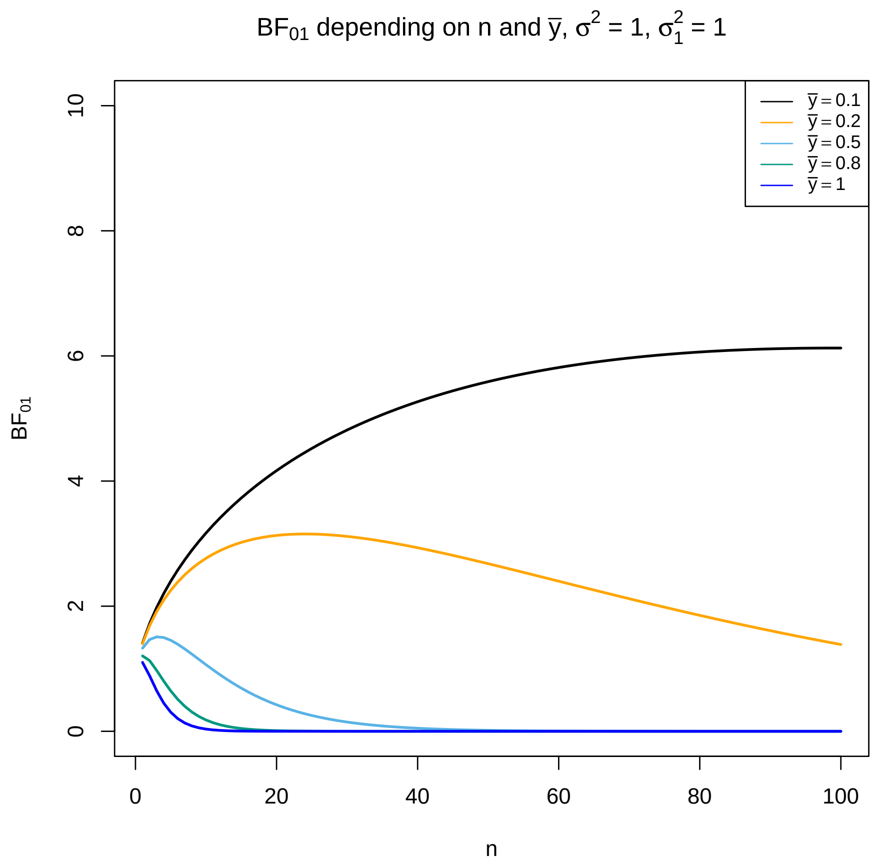


Figure 7: Comparing different means across different fixed parameters.

Proposition 2 (Influence of alternative prior width σ_1 on BF_{10}).
For $n \in [1, \infty), \sigma > 0, \sigma_1 > 0$:

1. *If $\sigma_1 > \bar{y}$ the function BF_{10} decreases strictly monotonically with respect to alternative prior width σ_1.*

2. *If $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$ the function BF_{10} increases strictly monotonically with respect to the alternative prior width σ_1.*

3. *As the alternative prior width σ_1 approaches its limits the following holds:*

$$\lim_{\sigma_1 \to 0} BF_{10} = 1 \text{ and } \lim_{\sigma_1 \to \infty} BF_{10} = 0.$$

The influence of the alternative prior width σ_1 on the Bayes Factor function BF_{10} is not trivial. There are two cases that are of interest $\sigma_1 > \bar{y}$ and $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$. All other cases are not relevant, because they have a non-monotonic influence or change monotonicity over the function.

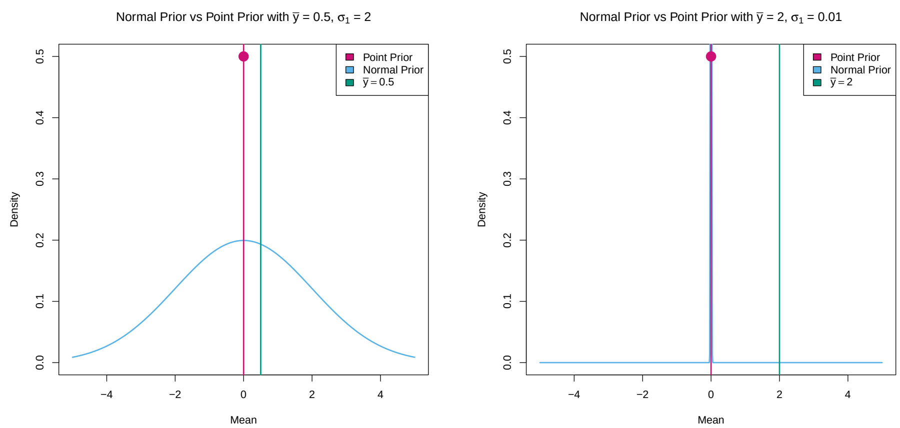


Figure 8: On the left side for $\bar{y} = 0.5, \sigma_1 = 2$, then $\sigma_1 > \bar{y}$: If σ_1 increases BF_{10} decreases towards a H_0 decision. On the right side for $\bar{y} = 2, \sigma_1 = 0.01$ with $\sigma^2 = 1$ then $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$: If σ_1 decreases, BF_{10} decreases towards an equal likeliness of hypotheses H_0 and H_1. Note that the alternative prior is very narrow and therefore barely visible.

Looking at the case where the alternative prior width is greater than the mean $\sigma_1 > \bar{y}$. If the alternative prior width σ_1 increases then BF_{10} decreases towards a H_0 decision. Intuitively, when the normal distribution is more spread out, the probability for the mean lying in the normal distribution of the alternative prior decreases, because it is already "inside" of the normal distribution. So misspecifying the alternative prior width by increasing it leads in an extreme case to an uniform distribution case, where the density is so spread out, that the point prior is the extremely likely model to decide for.
Now looking at the case where $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$. If the alternative prior width decreases, then BF_{10} decreases as well. Intuitively, when the normal distribution gets denser the probability of deciding for the H_1 also decreases, because the mean is already "outside" of the normal distribution. This leads in extreme cases to such a dense normal distribution that the normal prior is not distinguishable from the point prior anymore. Therefore this leads to an equal likeliness of hypotheses for H_0 and H_1. Also see Figure 8 for a visualisation.

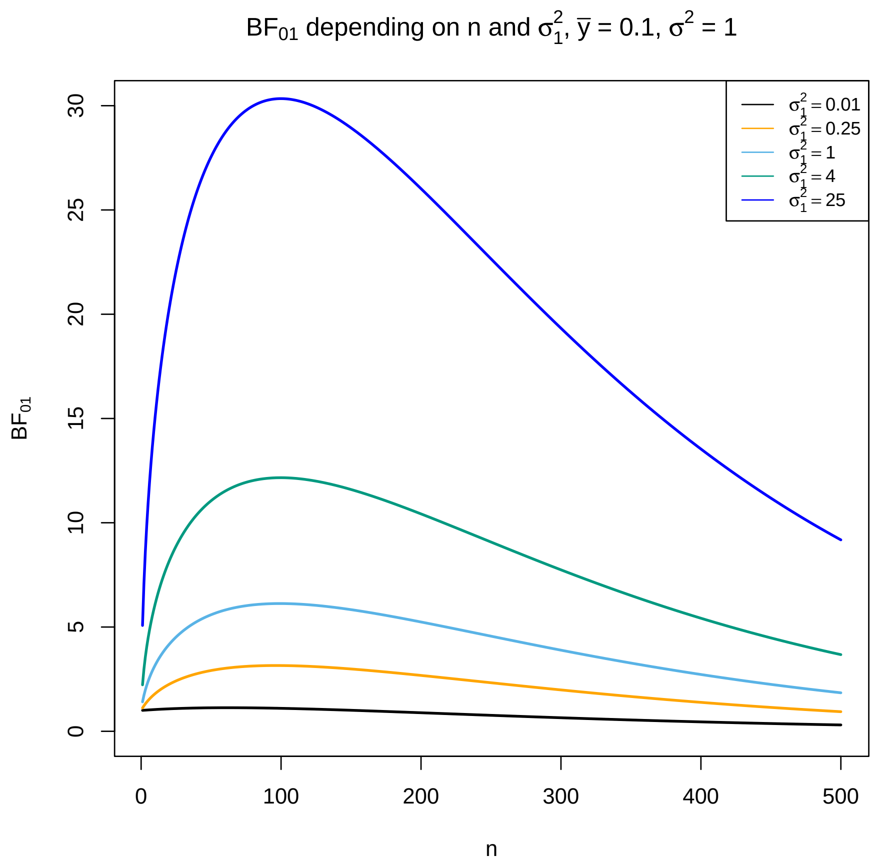


Figure 9: Comparing different alternative prior variances across fixed parameters.

Therefore it is of more interest to look into the case $\sigma_1 > \bar{y}$, because for the decreasing Bayes Factor towards a H_0 decision with respect to increasing σ_1. In Figure 9 there are different values of alternative prior width compared and visualised.

Proposition 3 (Newtons Method finds for $BF_{\mathbf{crit}}$ all global solutions in BF_{10}).
Given a critical Bayes Factor threshold BF_{crit} for any $\bar{y}, \sigma > 0, \sigma_1 > 0$ and a tolerance ε this algorithm will always find all correct solutions beyond that tolerance ε for $BF_{10}(\bar{y}, \sigma, \sigma_1) = BF_{crit}$ of which there are at most two: n_1 and n_2, if any solutions exist:

Input: $BF_{crit}, \bar{y}, \sigma > 0, \sigma_1 > 0, \varepsilon > 0$

Output: $\{n_1, n_2\}$ *or* $\{n_*\}$ *or* $\emptyset$

```
// No minimum exists
if (ȳ = 0) then
    n* = ε
    // Critical threshold greater than initial n
    if BF10(n*) < BFcrit then
        return ∅
    // Critical threshold smaller than or equal initial n
    else
        while BF10(n*) > BFcrit and n* > 0 do
            n* = newtonsMethod(n*, BFcrit, ȳ, σ, σ1)
        return {n*}
// A minimum exists
else
    n* = argMinBF10(ȳ, σ, σ1)
    // Minimum value at initial n intersects with critical
       threshold
    if (BFcrit = BF10(n*)) then
        return {n*}
    // Minimum value is greater than critical threshold
    else if BFcrit < BF10(n*) then
        return ∅
    // Minimum value is smaller than critical value
    else
        n1 = n* − ε
        n2 = n* + ε
        // Calculate left boundary
        while BF10(n1) < BFcrit and n1 > 0 do
            n1 = newtonsMethod(n1, BFcrit, ȳ, σ, σ1)
        // Calculate right boundary
        while BF10(n2) < BFcrit do
            n2 = newtonsMethod(n2, BFcrit, ȳ, σ, σ1)
        return {n1, n2}
```

Algorithm 3: Intersection points $n_{1/2}$ for $BF_{10} = BF_{\text{crit}}$

If one or multiple solutions are found, a transformation to the discrete space can be conducted as follows: if $|n_1 - \lfloor n_1 \rfloor| > |n_1 - \lceil n_1 \rceil|$, then $\lceil n_1 \rceil$ is chosen, else $\rfloor n_1 \lfloor$ is chosen. This is also the case for solution n_2 if it exists. Results of the algorithm are visualised in Figure 10.

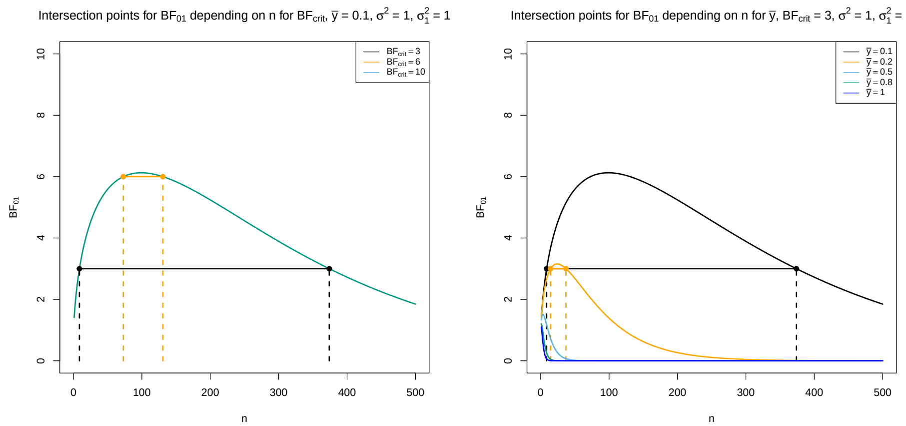


Figure 10: On the left Intersection points for different Bayes Factor thresholds BF_{crit} and on the right Intersection points for different true means $\bar{y}$.

3.1.2 Proofs

Before proving the propositions a few lemmas will be introduced. They are not proven here, but references to proofs are given below.

Lemma 1 (Product of two smooth functions is smooth).
For two smooth (infinitely differentiable) real functions $f(x)$ and $g(x)$ the product $f(x) \cdot g(x)$ is smooth.

Lemma 2 (Logarithm preserves order.).
The real logarithmic function $\ln(x)$ is smooth (infinitely differentiable) and strictly increasing for $x > 0$.

Lemma 3 (Partial derivative analysis for monotonic properties).
Given a real differentiable function $f(\tau, x_1, ..., x_n)$, where τ is the parameter of interest and $x_1, ..., x_n$ are other parameters. If the partial derivative $\frac{\partial f(\tau, x_1, ..., x_n)}{\partial \tau}$ with respect to τ is either strictly positive or strictly negative for all values of τ for any fixed parameters $x_1, ..., x_n$, then $f(\tau, x_1, ..., x_n)$ is strictly monotonic with respect to τ.

Lemma 4 (Newton's Method).
Given a real continuously differentiable function $f(x)$ and suppose there exists a root x_*, such that $f(x_*) = 0$ and $f'(x_*) \neq 0$. Given an initial guess x_0

sufficiently close to x_* and a tolerance $\varepsilon > 0$, such that $\varepsilon < |x_* - x_0|$ and applying Newtons Method with the iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $f'(x_{n+1}) \neq 0$ and ε was chosen small enough then $|x_* - x_{n+1}| < |x_* - x_n|$. Therefore x_{n+1} is a closer approximation to x_* than x_n. Repeating this method iteratively leads to a local convergence in x_* if $\varepsilon > |x_* - x_{n+1}|$ and x_* is a root of $f(x) = 0$.

For more information on Lemma 1, 2 and 3 look into an introduction book about mathematical analysis (e.g. "Principles of Mathematical Analysis" by Rudin W. or "Introduction to real analysis" by Bartle and Sherbert). For more information on Lemma 4 a look into a numerical analysis introduction book (e.g. "Numerical Analysis" by Burden and Faires) is advised.

Proposition 1 For $n \in (0, \infty), \sigma > 0, \sigma_1 > 0, \bar{y} \neq 0$ the function $BF_{10}(n, \bar{y}, \sigma, \sigma_1)$ (see Definition 1) always has a minimum argument at $\arg\min_n BF_{10}(n, \bar{y}, \sigma, \sigma_1) = n_*$ with

$$n_* = \sigma^2 \left(\frac{-2\bar{y}^2 + \sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2 \bar{y}^2} \right)$$

and a minimum $\min_n BF_{10}(n, \bar{y}, \sigma, \sigma_1) = BF_{10}(n_*, \bar{y}, \sigma, \sigma_1)$ with:

$$BF_{10}(n_*) = \frac{\bar{y}}{\sqrt{\frac{1}{2}\left(\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}\right)}} \cdot \exp\left[\frac{-\bar{y}^2 + \frac{1}{2}\sigma_1^2 + \sqrt{\bar{y}^4 + \frac{1}{4}\sigma_1^4} + \frac{2\bar{y}^4 - \bar{y}^2\sqrt{4\bar{y}^4 + \sigma_1^4}}{\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}} \right]$$

.

Proof. To prove this the following steps are conducted:

1. Calculate first derivation $\frac{\partial BF_{10}}{\partial n}$.
2. Solve $\frac{\partial BF_{10}}{\partial n} = 0$ for n.
3. Show there exists exactly one location of the assumed minimum $\arg\min_n BF_{10} = n_*$.

4. Calculate the second derivation via the logarithmic function $\ln(BF_{10})$ and show the extremum is indeed a minimum with $\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} > 0$
5. Calculate with the location n_* the minimum value $\min BF_{10} = BF_{10}(n_*)$.

Assume $n \in (0, \infty), \bar{y} \neq 0, \sigma > 0, \sigma_1 > 0$. Because of Lemma 1 as long as the denominator is not equal to 0 the function BF_{10} is smooth, because the exponential function and the rational are under those conditions smooth as well.

1. Calculate the first derivation $\frac{\partial BF_{10}}{\partial n}$.

$$
\begin{aligned}
BF_{10} &= \underbrace{\frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}}}_{f} \cdot \underbrace{\exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]}_{g} \\
\frac{\partial BF_{10}}{\partial n} &= \frac{\partial f}{\partial n} \cdot g + f \cdot \frac{\partial g}{\partial n} \\
f &= \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \\
\frac{\partial f}{\partial n} &= -\frac{1}{2}\sigma\sigma_1^2 \cdot (\sigma^2 + n\sigma_1^2)^{-\frac{3}{2}} \\
&= -\frac{\sigma\sigma_1^2}{2\sqrt{(\sigma^2 + n\sigma_1^2)^3}} \\
g &= \exp \underbrace{\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]}_{h} \\
h &= \frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} \\
\frac{\partial h}{\partial n} &= \frac{2n\sigma_1^2\bar{y}^2 \cdot 2\sigma^2(\sigma^2 + n\sigma_1^2) - (n^2\sigma_1^2\bar{y}^2 \cdot 2\sigma^2\sigma_1^2)}{(2\sigma^2(\sigma^2 + n\sigma_1^2))^2} \\
&= \frac{4n\sigma^4\sigma_1^2\bar{y}^2 + 4n^2\sigma^2\sigma_1^4\bar{y}^2 - 2n^2\sigma^2\sigma_1^4\bar{y}^2}{4\sigma^4(\sigma^2 + n\sigma_1^2)^2} \\
&= \frac{2n\sigma^4\sigma_1^2\bar{y}^2 + 2n^2\sigma^2\sigma_1^4\bar{y}^2 - n^2\sigma^2\sigma_1^4\bar{y}^2}{2\sigma^4(\sigma^2 + n\sigma_1^2)^2} \\
&= \frac{2n\sigma^4\sigma_1^2\bar{y}^2 + n^2\sigma^2\sigma_1^4\bar{y}^2}{2\sigma^4(\sigma^2 + n\sigma_1^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2n\sigma^2\sigma_1^2\bar{y}^2 + n^2\sigma_1^4\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)^2} \\
&= \frac{n\sigma_1^2\bar{y}^2(2\sigma^2 + n\sigma_1^2)}{2\sigma^2(\sigma^2 + n\sigma_1^2)^2} \\
\frac{\partial g}{\partial n} &= \frac{\partial \exp(h)}{\partial n} = \frac{\partial h}{\partial n}\exp(h) \\
&= \frac{n\sigma_1^2\bar{y}^2(2\sigma^2 + n\sigma_1^2)}{2\sigma^2(\sigma^2 + n\sigma_1^2)^2} \cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BF_{10}}{\partial n} &= -\frac{\sigma\sigma_1^2}{2\sqrt{(\sigma^2 + n\sigma_1^2)^3}} \cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&+ \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \frac{n\sigma_1^2\bar{y}^2(2\sigma^2 + n\sigma_1^2)}{2\sigma^2(\sigma^2 + n\sigma_1^2)^2} \cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \left(-\frac{\sigma\sigma_1^2}{2(\sigma^2 + n\sigma_1^2)^{3/2}} + \frac{n\sigma_1^2\bar{y}^2(2\sigma^2 + n\sigma_1^2)}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}}\right) \\
&\cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \left(-\frac{\sigma^2\sigma_1^2(\sigma^2 + n\sigma_1^2)}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}} + \frac{n\sigma_1^2\bar{y}^2(2\sigma^2 + n\sigma_1^2)}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}}\right) \\
&\cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \sigma_1^2\left(\frac{-\sigma^4 - n\sigma^2\sigma_1^2 + 2n\sigma^2\bar{y}^2 + n^2\sigma_1^2\bar{y}^2}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}}\right) \cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]
\end{aligned}
$$

2. Solve $\frac{\partial BF_{10}}{\partial n} = 0$ for n.

$$
\begin{aligned}
\frac{\partial BF_{10}}{\partial n} &= 0 \\
&= \sigma_1^2\left(\frac{-\sigma^4 - n\sigma^2\sigma_1^2 + 2n\sigma^2\bar{y}^2 + n^2\sigma_1^2\bar{y}^2}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}}\right) \cdot \exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \underbrace{\sigma_1^2\left(\frac{-\sigma^4 - n\sigma^2\sigma_1^2 + 2n\sigma^2\bar{y}^2 + n^2\sigma_1^2\bar{y}^2}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}}\right)}_{\text{Zero Product Property } zpp} \cdot \underbrace{\exp\left[\frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]}_{\text{never becomes 0}}
\end{aligned}
$$

$$
\begin{aligned}
zpp &= 0\\
&= -\sigma^4 - n\sigma^2\sigma_1^2 + 2n\sigma^2\bar{y}^2 + n^2\sigma_1^2\bar{y}^2\\
&= n^2\sigma_1^2\bar{y}^2 + n(2\sigma^2\bar{y}^2 - \sigma^2\sigma_1^2) - \sigma^4\\
&= n^2\underbrace{\sigma_1^2\bar{y}^2}_{a} + n\underbrace{(2\sigma^2\bar{y}^2 - \sigma^2\sigma_1^2)}_{b}\underbrace{-\sigma^4}_{c}\\
&= \underbrace{an^2 + bn + c}_{\text{Quadratic formula}}\\
n_{1/2} &= \frac{-(2\sigma^2\bar{y}^2 - \sigma^2\sigma_1^2) \pm \sqrt{(2\sigma^2\bar{y}^2 - \sigma^2\sigma_1^2)^2 - 4(\sigma_1^2\bar{y}^2)(-\sigma^4)}}{2\sigma_1^2\bar{y}^2}\\
&= \frac{-2\sigma^2\bar{y}^2 + \sigma^2\sigma_1^2 \pm \sqrt{4\sigma^4\bar{y}^4 - 4\sigma^4\sigma_1^4\bar{y}^2 + \sigma^4\sigma_1^4 + 4\sigma^4\sigma_1^2\bar{y}^2}}{2\sigma_1^2\bar{y}^2}\\
&= \sigma^2\left(\frac{-2\bar{y}^2 + \sigma_1^2 \pm \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)
\end{aligned}
$$

3. Show there exists exactly one location of the assumed minimum $\arg\min_n BF_{10} = n_*$.

   $n_1 \in (0, \infty)$:

$$
\begin{aligned}
n_1 &= \frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\\
\text{For } \sigma_1 \neq 0 \text{ and } \bar{y} \neq 0 &:\\
\sigma_1^2 &> 0\\
\sigma_1^2 &> 2\bar{y}^2 - 2\bar{y}^2\\
\text{Because } -2\bar{y}^2 &= -\sqrt{4\bar{y}^4} > -\sqrt{4\bar{y}^4 + \sigma_1^4}\\
\sigma_1^2 &> 2\bar{y}^2 - \sqrt{4\bar{y}^4 + \sigma_1^4}\\
\frac{\sigma_1^2}{2\bar{y}^2\sigma_1^2} &> \frac{2\bar{y}^2 - \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\\
\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2} &> 0\\
\sigma^2 \cdot \left(\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\right) &> 0\\
n_1 &> 0
\end{aligned}
$$

$n_2 \in (-\infty, 0)$:

$$
\begin{aligned}
n_2 &= \frac{\sigma_1^2 - 2\bar{y}^2 - \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2} \\
\text{For } \sigma_1 \neq 0 \text{ and } \bar{y} \neq 0 : \\
-2\bar{y}^2 &< 0 \\
0 &< 2\bar{y}^2 \\
\sigma_1^2 &< 2\bar{y}^2 + \sigma_1^2 \\
\text{Because } \sigma_1^2 &= \sqrt{\sigma_1^4} < \sqrt{4\bar{y}^4 + \sigma_1^2} \\
\frac{\sigma_1^2}{2\bar{y}^2\sigma_1^2} &< \frac{2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2} \\
\frac{\sigma_1^2 - 2\bar{y}^2 - \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2} &< 0 \\
\sigma^2 \cdot \left(\frac{\sigma_1^2 - 2\bar{y}^2 - \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2} \right) &< 0 \\
n_2 &< 0
\end{aligned}
$$

Therefore the only extrema is at $n_* = n_1 = (0, \infty)$.

4. Calculate the second derivation via the logarithmic function $\ln(BF_{10})$ and show the extremum is indeed a minimum with $\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} > 0$

$$
\begin{aligned}
\ln(BF_{10}) &= \ln\left(\frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} \right] \right) \\
&= \ln\left(\frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \right) + \ln\left(\exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} \right] \right) \\
&= \ln\sigma - \ln\left(\sqrt{\sigma^2 + n\sigma_1^2} \right) + \frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} \\
&= \frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} - \ln\left(\sqrt{\sigma^2 + n\sigma_1^2} \right) + \ln\sigma \\
&= \frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)} - \frac{1}{2}\ln\left(\sigma^2 + n\sigma_1^2 \right) + \ln\sigma
\end{aligned}
$$

It is important to note that after Lemma 2, the properties of the logarithmic function preserve the order, if all the inputs are defined for it. Therefore determining a minimum with the second derivative of the logarithmic function $\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} > 0$ still holds for the original term $\frac{\partial^2 BF_{10}}{\partial^2 n} > 0$. Thus, this is the resulting function definition:

$$\ln(BF_{10}) : \mathbb{R}_+ \to R : n \mapsto \frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{1}{2}\ln\left(\sigma^2+n\sigma_1^2\right) + \ln\sigma$$

Calculation of the first derivation $\frac{\partial \ln BF_{10}}{\partial n}$:

$$\begin{aligned}
\frac{\partial \ln(BF_{10})}{\partial n} &= \frac{2n\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)} - 2\sigma^2\sigma_1^2\frac{n^2\sigma_1^2\bar{y}^2}{(2\sigma^2(\sigma^2+n\sigma_1^2))^2} - \frac{\sigma_1^2}{2(\sigma^2+n\sigma_1^2)} \\
&= \frac{n\sigma_1^2\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{n^2\sigma_1^4\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{\sigma_1^2}{2(\sigma^2+n\sigma_1^2)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} &= \frac{\sigma_1^2\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{n\sigma_1^4\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n\sigma_1^4\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} \\
&+ \frac{2n^2\sigma_1^6\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3} + \frac{2\sigma_1^4}{(2\sigma^2+n\sigma_1^2)^2} \\
&= \frac{\sigma_1^2\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{2n\sigma_1^4\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} + \frac{n^2\sigma_1^6\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^3} \\
&+ \frac{\sigma_1^4}{2(\sigma^2+n\sigma_1^2)^2} \\
&= \frac{2\sigma_1^2\bar{y}^2(\sigma^2+n\sigma_1^2) - 4n\sigma_1^4\bar{y}^2 + \sigma^2\sigma_1^4}{2\sigma^2(\sigma^2+n\sigma_1^2)^2} + \frac{2n^2\sigma_1^6\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3} \\
&= \frac{2\sigma^2\sigma_1^2\bar{y}^2 + 2n\sigma_1^4\bar{y}^2 - 4n\sigma_1^4\bar{y}^2 + \sigma^2\sigma_1^4}{2\sigma^2(\sigma^2+n\sigma_1^2)^2} \\
&+ \frac{2n^2\sigma_1^6\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3} \\
&= \frac{2\sigma^2\sigma_1^2\bar{y}^2 - 2n\sigma_1^4\bar{y}^2 + \sigma^2\sigma_1^4}{2\sigma^2(\sigma^2+n\sigma_1^2)^2} + \frac{2n^2\sigma_1^6\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3} \\
&= \sigma_1^2 \cdot \left(\frac{2\sigma^2\bar{y}^2 - 2n\sigma_1^2\bar{y}^2 + \sigma^2\sigma_1^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^2} + \frac{2n^2\sigma_1^4\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3}\right) \\
&= \sigma_1^2 \cdot \left(\frac{(2\sigma^2\bar{y}^2 - 2n\sigma_1^2\bar{y}^2 + \sigma^2\sigma_1^2)(\sigma^2+n\sigma_1^2) + 2n^2\sigma_1^4\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)^3}\right)
\end{aligned}$$

$$= \sigma_1^2 \cdot \left(\frac{2\sigma^4 \bar{y}^2 - 2n\sigma^2\sigma_1^2\bar{y}^2 + \sigma^4\sigma_1^2 + 2n\sigma^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)^3} - \frac{-2n^2\sigma_1^4\bar{y}^2 + n\sigma^2\sigma_1^4 + 2n^2\sigma_1^4\bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)^3} \right)$$

$$= \sigma^2\sigma_1^2 \cdot \left(\frac{2\sigma^2\bar{y}^2 + \sigma^2\sigma_1^2 + n\sigma_1^4}{2\sigma^2(\sigma^2 + n\sigma_1^2)^3} \right)$$

$$= \sigma_1^2 \cdot \left(\frac{2\sigma^2\bar{y}^2 + \sigma^2\sigma_1^2 + n\sigma_1^4}{2(\sigma^2 + n\sigma_1^2)^3} \right)$$

Therefore it is left to prove, that $\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} > 0$:

$$\text{For } \sigma > 0 \text{ and } \sigma_1 > 0 :$$

$$\underbrace{2\sigma^2\bar{y}^2}_{\geq 0} + \underbrace{n\sigma_1^4}_{>0} + \underbrace{\sigma^2\sigma_1^2}_{>0} > 0$$

$$\sigma_1^2 \cdot (2\sigma^2\bar{y}^2 + n\sigma_1^4 + \sigma^2\sigma_1^2) > 0$$

$$\sigma_1^2 \cdot \left(\frac{2\sigma^2\bar{y}^2 + n\sigma_1^4 + \sigma^2\sigma_1^2}{2(\sigma^2 + n\sigma_1^2)^3} \right) > 0$$

$$\frac{\partial^2 \ln(BF_{10})}{\partial^2 n} > 0$$

Therefore all extrema at n_* are always minima.

5. Calculate with the location n_* the minimum value $\min BF_{10} = BF_{10}(n_*)$.

$$BF_{10}(n_*) = \frac{\sigma}{\sqrt{\sigma^2 + \sigma^2 \left(\frac{-2\bar{y}^2 + \sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2\bar{y}^2} \right) \sigma_1^2}}$$

$$\cdot \exp \left[\frac{\left(\sigma^2 \left(\frac{-2\bar{y}^2 + \sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2\bar{y}^2} \right) \right)^2 \bar{y}^2 \sigma_1^2}{2\sigma^2 \left(\sigma^2 + \sigma^2 \left(\frac{-2\bar{y}^2 + \sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\sigma_1^2\bar{y}^2} \right) \sigma_1^2 \right)} \right]$$

$$= \frac{\sigma}{\sigma\sqrt{1+\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)\sigma_1^2}}$$

$$\cdot \exp\left[\frac{\sigma^4\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)^2\bar{y}^2\sigma_1^2}{2\sigma^4\left(1+\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)\sigma_1^2\right)}\right]$$

$$= \frac{1}{\sqrt{1+\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)\sigma_1^2}}$$

$$\cdot \exp\left[\frac{\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)^2\bar{y}^2\sigma_1^2}{2\left(1+\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)\sigma_1^2\right)}\right]$$

$$= \frac{1}{\sqrt{1+\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\bar{y}^2}}}$$

$$\cdot \exp\left[\frac{\left(\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\sigma_1^2\bar{y}^2}\right)^2\bar{y}^2\sigma_1^2}{2+\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{\bar{y}^2}}\right]$$

$$= \frac{1}{\sqrt{1+\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{2\bar{y}^2}}}$$

$$\cdot \exp\left[\frac{\left(\frac{8\bar{y}^4-4\sigma_1^2\bar{y}^2-4\bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4}+2\sigma_1^4+2\sigma_1^2\sqrt{4\bar{y}^4+\sigma_1^4}}{4\sigma_1^4\bar{y}^4}\right)\bar{y}^2\sigma_1^2}{2+\frac{-2\bar{y}^2+\sigma_1^2+\sqrt{4\bar{y}^4+\sigma_1^4}}{\bar{y}^2}}\right]$$

$$= \frac{1}{\sqrt{1 - 1 + \frac{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2}}}$$

$$\cdot \exp\left[\frac{\frac{8\bar{y}^4 - 4\sigma_1^2\bar{y}^2 - 4\bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4} + 2\sigma_1^4 + 2\sigma_1^2\sqrt{4\bar{y}^4+\sigma_1^4}}{4\sigma_1^2\bar{y}^2}}{2 - 2 + \frac{\sigma_1^2 + \sqrt{4\bar{y}^4+\sigma_1^4}}{\bar{y}^2}}\right]$$

$$= \frac{1}{\sqrt{\frac{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2}}}$$

$$\cdot \exp\left[\frac{\frac{8\bar{y}^4 - 4\sigma_1^2\bar{y}^2 - 4\bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4} + 2\sigma_1^4 + 2\sigma_1^2\sqrt{4\bar{y}^4+\sigma_1^4}}{4\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4+\sigma_1^4}}\right]$$

$$= \frac{1}{\sqrt{\frac{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2}}}$$

$$\cdot \exp\left[\frac{-\bar{y}^2 + \frac{1}{2}\sigma_1^2 + \sqrt{\bar{y}^4 + \frac{1}{4}\sigma_1^4} + \frac{2\bar{y}^4 - \bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4}}{\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4+\sigma_1^4}}\right]$$

$$= \frac{\bar{y}}{\sqrt{\frac{1}{2}\left(\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}\right)}}$$

$$\cdot \exp\left[\frac{-\bar{y}^2 + \frac{1}{2}\sigma_1^2 + \sqrt{\bar{y}^4 + \frac{1}{4}\sigma_1^4} + \frac{2\bar{y}^4 - \bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4}}{\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4+\sigma_1^4}}\right]$$

Therefore, there exists only one minimum with

$$\arg\min_n BF_{10} = \sigma^2\left(\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\right)$$

and

$$\min BF_{10} = \frac{\bar{y}}{\sqrt{\frac{1}{2}\left(\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}\right)}}$$

$$\cdot \exp\left[\frac{-\bar{y}^2 + \frac{1}{2}\sigma_1^2 + \sqrt{\bar{y}^4 + \frac{1}{4}\sigma_1^4} + \frac{2\bar{y}^4 - \bar{y}^2\sqrt{4\bar{y}^4+\sigma_1^4}}{\sigma_1^2}}{\sigma_1^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}\right]$$

□

Corollar 1 For the function BF_{10} the point of the minimum $\arg\min_n BF_{10}$ scales quadratically with the true standard deviation σ. The value of the minimum $\min_n BF_{10}$ is independent of the true standard deviation σ.

Proof. The proof is divided in two parts:

1. Prove proportionality of σ^2 on n_*, such that $n_{*,\sigma} = \sigma^2 \cdot n_{*,\sigma=1}$:

$$\begin{aligned} n_{*,\sigma} &= \sigma^2 \left(\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\right) \\ &= \sigma^2 \cdot 1 \cdot \left(\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2\sigma_1^2}\right) \\ &= \sigma^2 \cdot n_{*,\sigma=1} \end{aligned}$$

2. Prove independency of σ on $BF_{10}(n_*)$ for $s, t \in \mathbb{R}_+$, such that $BF_{10,\sigma=s}(n_*) = BF_{10,\sigma=t}(n_*)$: This follows directly from Proposition 1 .

□

Corollar 2 For $n \in (0,\infty), \sigma > 0, \sigma_1 > 0$ the function BF_{10} increases strictly monotonically with respect to $|\bar{y}|$ for all means $\bar{y} \neq 0$.

Proof. Let $n \in (0,\infty), \sigma > 0, \sigma_1 > 0$. With Lemma 3 strictly increasing monotonicity for all positive $\bar{y}$ is proven by taking the partial derivative $\frac{\partial BF_{10}}{\partial \bar{y}}$ and checking if it is strictly positive for all $\bar{y}$ for any other fixed parameter. The logarithmic function property of Lemma 2 for preserving the order and the composition of the logarithmic function ln with the Bayes Factor function BF_{10}

from Proposition 1 will be used.

Therefore the conditions for $\frac{\partial \ln(BF_{10})}{\partial \bar{y}} > 0$ or $\frac{\partial \ln(BF_{10})}{\partial \bar{y}} < 0$ will be proven:

$$\begin{aligned} \ln(BF_{10}) &= \frac{n^2\sigma_1^2\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{1}{2}\ln\left(\sigma^2+n\sigma_1^2\right) + \ln\sigma \\ \frac{\partial \ln(BF_{10})}{\partial \bar{y}} &= \bar{y}\cdot\frac{2n^2\sigma_1^2}{2\sigma^2(\sigma^2+n\sigma_1^2)} \end{aligned}$$

Without loss of generality:

$$\begin{aligned} \bar{y} &> 0 \\ \bar{y}\cdot\frac{2n^2\sigma_1^2}{2\sigma^2(\sigma^2+n\sigma_1^2} &> 0 \\ \frac{\partial \ln(BF_{10})}{\partial \bar{y}} &> 0 \end{aligned}$$

For all $\bar{y} > 0$ the function BF_{10} is strictly monotonically increasing in respect to $\bar{y}$. Analogously for all $\bar{y} < 0$ the function BF_{10} is strictly monotonically decreasing in respect to $\bar{y}$. Therefore the function BF_{10} is strictly monotonically increasing in respect to $|\bar{y}|$. □

Proposition 2 For $n \in [1,\infty), \sigma > 0, \sigma_1 > 0$:

1. If $\sigma_1 > \bar{y}$ the function BF_{10} decreases strictly monotonically with respect to alternative prior width σ_1.

2. If $\sigma_1 < \sqrt{\bar{y}^2-\sigma^2}$ the function BF_{10} increases strictly monotonically with respect to the alternative prior width σ_1.

3. As the alternative prior width σ_1 approaches its limits the following holds:

$$\lim_{\sigma_1\to 0} BF_{10} = 1 \text{ and } \lim_{\sigma_1\to\infty} BF_{10} = 0.$$

Proof. Let $n \in (0,\infty), \sigma > 0, \sigma_1 > 0$. With Lemma 3 strictly increasing monotonicity for all positive σ_1 is proven by taking the partial derivative $\frac{\partial BF_{10}}{\partial \sigma_1}$ and look, if it is strictly positive for all σ_1 for any other fixed parameter. The logarithmic function property of Lemma 2 for preserving the order and the composition of the logarithmic function ln with the Bayes Factor function BF_{10} from Proposition 1 will be used.

Therefore the conditions for $\frac{\partial \ln(BF_{10})}{\partial \sigma_1} > 0$ or $\frac{\partial \ln(BF_{10})}{\partial \sigma_1} < 0$ will be proven:

$$\begin{aligned}
\frac{\partial \ln(BF_{10})}{\partial \sigma_1} &= \frac{2n^2\sigma_1\bar{y}^2}{2\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{(4n\sigma^2\sigma_1)n^2\sigma_1^2\bar{y}^2}{(2\sigma^2(\sigma^2+n\sigma_1^2))^2} - \frac{n\sigma_1}{\sigma^2+n\sigma_1^2} \\
&= \frac{n^2\sigma_1\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)} - \frac{n^3\sigma_1^3\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n\sigma_1}{\sigma^2+n\sigma_1^2} \\
&= \frac{n^2\sigma_1\bar{y}^2(\sigma^2+n\sigma_1^2)}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n^3\sigma_1^3\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n\sigma^2\sigma_1(\sigma^2+n\sigma_1^2)}{\sigma^2(\sigma^2+n\sigma_1^2)^2} \\
&= \frac{n^2\sigma^2\sigma_1\bar{y}^2+n^3\sigma_1^3\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n^3\sigma_1^3\bar{y}^2}{\sigma^2(\sigma^2+n\sigma_1^2)^2} - \frac{n\sigma^4\sigma_1+n^2\sigma^2\sigma_1^3}{\sigma^2(\sigma^2+n\sigma_1^2)^2} \\
&= \frac{n^2\sigma^2\sigma_1\bar{y}^2 - n\sigma^4\sigma_1 - n^2\sigma^2\sigma_1^3}{\sigma^2(\sigma^2+n\sigma_1^2)^2} \\
&= \frac{n^2\sigma_1\bar{y}^2 - n\sigma^2\sigma_1 - n^2\sigma_1^3}{(\sigma^2+n\sigma_1^2)^2} \\
&= n\sigma_1 \cdot \left(\frac{n\bar{y}^2-\sigma^2-n\sigma_1^2}{(\sigma^2+n\sigma_1^2)^2}\right)
\end{aligned}$$

There are two cases relevant for the monotonicity for the whole function BF_{10}:

1. Case: $\sigma_1 > \bar{y}$

$$\begin{aligned}
\sigma_1 > \bar{y} &\geq \sqrt{\bar{y}^2 - \frac{\sigma^2}{n}} \\
\sigma_1^2 &> \bar{y}^2 - \frac{\sigma^2}{n} \\
0 &> \bar{y}^2 - \frac{\sigma^2}{n} - \sigma_1^2 \\
0 &> n\bar{y}^2 - \sigma^2 - n\sigma_1^2 \\
0 &> \frac{n\bar{y}^2 - \sigma^2 - n\sigma_1^2}{\sigma^2(\sigma^2 + n\sigma_1^2)^2} \\
0 &> n\sigma_1 \cdot \left(\frac{n\bar{y}^2 - \sigma^2 - n\sigma_1^2}{\sigma^2(\sigma^2 + n\sigma_1^2)^2}\right) \\
0 &> \frac{\partial \ln(BF_{10})}{\partial \sigma_1}
\end{aligned}$$

Note, if $\sigma_1 > \bar{y}$ yields, in respect to the alternative prior width the whole function is monotonically decreasing, even for large n, especially for

$$\sigma_1 > \lim_{n\to\infty} \sqrt{\bar{y}^2 - \frac{\sigma^2}{n}} = \sqrt{\bar{y}^2 - 0} = \bar{y}$$

2. Case: $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$

$$\begin{aligned}
\sigma_1 &< \sqrt{\bar{y}^2 - \sigma^2} \leq \sqrt{\bar{y}^2 - \frac{\sigma^2}{n}} \\
\sigma_1^2 &< \bar{y}^2 - \frac{\sigma^2}{n} \\
0 &< \bar{y}^2 - \frac{\sigma^2}{n} - \sigma_1^2 \\
0 &< n\bar{y}^2 - \sigma^2 - n\sigma_1^2 \\
0 &< \frac{n\bar{y}^2 - \sigma^2 - n\sigma_1^2}{\sigma^2(\sigma^2 + n\sigma_1^2)^2} \\
0 &< n\sigma_1 \cdot \left(\frac{n\bar{y}^2 - \sigma^2 - n\sigma_1^2}{\sigma^2(\sigma^2 + n\sigma_1^2)^2}\right) \\
0 &< \frac{\partial \ln(BF_{10})}{\partial \sigma_1}
\end{aligned}$$

Note, if $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$ yields, in respect to the alternative prior width the whole function is monotonically decreasing, even for small n. Since there is an especial interest in $n \geq 1$, this yields for

$$\sigma_1 < \lim_{n \to 1} \sqrt{\bar{y}^2 - \frac{\sigma^2}{n}} = \sqrt{\bar{y}^2 - \sigma^2}$$

Therefore it is proven, that given $\sigma_1 > \bar{y}$, the function BF_{10} (Lemma 2) in respect to σ_1 is strictly monotonically decreasing. However, as long as $\sigma_1 < \sqrt{\bar{y}^2 - \frac{\sigma}{n}}$ holds, the function BF_{10} in respect to σ_1 is strictly monotonically increasing.

3. Proof for $\lim_{\sigma_1 \to 0} BF_{10} = 1$ and $\lim_{\sigma_1 \to \infty} BF_{10} = 0$:

$$\begin{aligned}
\lim_{\sigma_1 \to 0} BF_{10} &= \lim_{\sigma_1 \to 0} \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \frac{\sigma}{\sqrt{\sigma^2 + 0}} \cdot \lim_{\sigma_1 \to 0} \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= 1 \cdot \exp\left[\frac{n^2\bar{y}^2 \cdot 0}{2\sigma^2(\sigma^2 + 0)}\right] \\
&= 1 \cdot \exp[0] \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{\sigma_1 \to \infty} BF_{10} &= \lim_{\sigma_1 \to \infty} \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \exp\left[\frac{n^2\bar{y}^2\sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right] \\
&= \lim_{\sigma_1 \to \infty} \frac{\sigma_1 \frac{\sigma}{\sigma_1}}{\sigma_1 \sqrt{\frac{\sigma^2}{\sigma_1^2} + n}} \cdot \exp\left[\frac{\sigma_1^2 n^2 \bar{y}^2}{\sigma_1^2 2\sigma^2(\frac{\sigma^2}{\sigma_1^2} + n)}\right] \\
&= \lim_{\sigma_1 \to \infty} \frac{\frac{\sigma}{\sigma_1}}{\sqrt{\frac{\sigma^2}{\sigma_1^2} + n}} \cdot \exp\left[\frac{n^2\bar{y}^2}{2\sigma^2(\frac{\sigma^2}{\sigma_1^2} + n)}\right] \\
&= \frac{0}{\sqrt{0 + n}} \cdot \exp\left[\frac{n^2\bar{y}}{2\sigma^2 n}\right] \\
&= 0 \cdot \exp\left[\frac{n\bar{y}^2}{2\sigma^2}\right] \\
&= 0
\end{aligned}$$

□

Proposition 3 Given a critical Bayes Factor threshold BF_{crit} for any $\bar{y}, \sigma > 0, \sigma_1 > 0$ and a tolerance ε *Algorithm* 3 will always find all correct solutions beyond that tolerance ε for $BF_{10}(\bar{y}, \sigma, \sigma_1) = BF_{\text{crit}}$ of which there are at most two: n_1 and n_2, if any solutions exist.

Proof. For $BF_{\text{crit}} \in (0, 1), \sigma > 0, \sigma_1 > 0, \varepsilon > 0$ the proposed algorithm calculates the correct result by using Newtons Method after Lemma 4. Following functions are used in the algorithm:

Input: $n, \bar{y}, \sigma, \sigma_1$

Output: n_*

$n_* \leftarrow \sigma^2 \cdot \left(\frac{\sigma_1^2 - 2\bar{y}^2 + \sqrt{4\bar{y}^4 + \sigma_1^4}}{2\bar{y}^2 \sigma_1^2} \right)$

return n_*

Function argMinBF10($n, \bar{y}, \sigma, \sigma_1$)

Input: $n, \bar{y}, \sigma, \sigma_1$

Output: BF10

BF10 $\leftarrow \frac{\sigma}{\sqrt{\sigma^2 + n\sigma_1^2}} \cdot \exp\left[\frac{n^2 \bar{y}^2 \sigma_1^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]$

return BF10

Function BF10($n, \bar{y}, \sigma, \sigma_1$)

Input: $n, \bar{y}, \sigma, \sigma_1$

Output: $\frac{\partial BF_{10}}{\partial n}(n, \bar{y}, \sigma, \sigma_1)$

derBF10 $\leftarrow \sigma_1^2 \left(\frac{-\sigma^4 - n\sigma^2\sigma_1^2 + 2n\sigma^2\bar{y}^2 + n^2\sigma_1^2\bar{y}^2}{2\sigma(\sigma^2 + n\sigma_1^2)^{5/2}} \right) \cdot \exp\left[\frac{n^2 \sigma_1^2 \bar{y}^2}{2\sigma^2(\sigma^2 + n\sigma_1^2)}\right]$

return derBF10

Function derBF10($n, \bar{y}, \sigma, \sigma_1$)

Input: $n_i, \bar{y}, \sigma, \sigma_1$

Output: n_{i+1}

$n_{i+1} \leftarrow n_i - \frac{\text{BF10}(n_i, \bar{y}, \sigma, \sigma_1)}{\text{derBF10}(n_i, \bar{y}, \sigma, \sigma_1)}$

return n_{i+1}

Function newtonsMethod($n_i, \bar{y}, \sigma, \sigma_1$)

The function argMinBF10 is implemented directly from the $\arg\min_n BF_{10}$ in Proposition 1. The function BF10 follows directly from Definition 1, the function derBF10 is the implementation of the derivative of BF_{10}. Its correctness is proven in the proof of Proposition 1. Finally the function newtonsMethod is implemented after Newtons Method, specified in Lemma 4. Note, that only the iterative process of Newtons Method is implemented in this function.

There are three cases to be proven in regards of correctness of the algorithm:

Case 1: $\bar{y} = 0$

For $\bar{y} = 0$ there exists no minimum in BF_{10}. Therefore there is either no intersection point with $BF_{\text{crit}} = BF_{10}$ or there is exactly one. However for large n:

$$
\begin{aligned}
\lim_{n\to\infty} BF_{10}(\bar{y}=0,n,\sigma,\sigma_1) &= \lim_{n\to\infty}\left(\frac{\sigma}{\sqrt{\sigma^2+n\sigma_1^2}}\cdot\exp\left[\frac{0}{2\sigma^2(\sigma^2+n\sigma_1^2)}\right]\right)\\
&= \lim_{n\to\infty}\frac{\sqrt{n}\frac{\sigma}{\sqrt{n}}}{\sqrt{n}\sqrt{\frac{\sigma^2}{n}+\sigma_1^2}}\\
&= \frac{0}{\sqrt{0+\sigma_1^2}}\\
&= 0
\end{aligned}
$$

It is easy to see, that for $\bar{y}=0$ the function BF_{10} is strictly monotonic decreasing in respect to n as the exponential term falls away.

If $BF_{10} < BF_{\text{crit}}$ at $n=\varepsilon$ there is no intersection point. Otherwise there is an intersection point, that can be found by using Newtons Method (see Lemma 4) with initial point $n_* = \varepsilon$.

Case 2: $\bar{y} \neq 0$ and $BF_{\text{crit}} = \min BF_{10}$

If $\bar{y}\neq 0$, a minimum exists at $(n_*, BF_{10}(n_*))$. Because $BF_{\text{crit}} = BF_{10}(n_*)$, there exists only one intersection point at $(n_*, BF_{10}(n_*))$. Therefore the chosen initial point n_* is the solution.

Case 3: $\bar{y} \neq 0$ and $BF_{\text{crit}} \neq \min BF_{10}$

If $\bar{y}\neq 0$, a minimum exists at $(n_*, BF_{10}(n_*))$. Either $\min BF_{10}(n_*) > BF_{\text{crit}}$, then there is no intersection point as the critical threshold is lower than the minimum of BF_{10}, or $\min BF_{10}(n_*) < BF_{\text{crit}}$, then there exist two intersection points n_1 and n_2. These points can be found by using Newtons method if they are adjusted by ε, such that $n_1 = n_* - \varepsilon$ and $n_2 = n_* + \varepsilon$. Therefore the assumption $\frac{\partial BF_{10}}{\partial n}(n_1) \neq 0 \neq \frac{\partial BF_{10}}{\partial n}(n_2)$ holds and Newtons Method can be applied (after Lemma 4) iteratively for n_1 and n_2 to find the correct solutions.

Note that especially for the cases in which the minimum exists: the initial $n_{1/2}$ are always chosen with $n_1 = n_* - \varepsilon$ and $n_2 = n_* + \varepsilon$ of the $\arg\min_n BF_{10} = n_*$. If ε is continuously decreasing all solutions can be found accordingly to the algorithm. Therefore the algorithm calculates the correct solutions for $BF_{10} = BF_{\text{crit}}$ in respect to $n_{1/2}$, if any solution exists. □

3.2 Optional Stopping Simulation

The Optional Stopping simulation is divided into three parts: Firstly, the Optional Stopping results based on the idealised setting for the asymmetrical and the symmetrical stopping case the are reported in Section 3.2.1. Secondly, an approximation for the asymmetrical and symmetrical case in the idealised setting will be derived in Section 3.2.2. Finally, the Optional Stopping results based on the realistic setting for the symmetrical stopping case are reported in Section 3.2.3.

3.2.1 Normal Prior With Known Variance (Idealised Setting)

The Asymmetrical Optional Stopping Procedure stops when a calculated Bayes Factor based on the collected data is smaller than the critical Bayes Factor decision threshold for H_0.

The results for the asymmetrical Optional Stopping simulation with all 20000 repetitions are shown in Figure 11.

The decision probability $P('H_0')$ was calculated by taking the decision count of Count($'H_0'$) and dividing it by the sum of the decision count of Count($'H_0'$) and Count($'H_{1\text{ or indecisive}}'$), such that $P('H_0') = \frac{\text{Count}('H_0')}{\text{Count}('H_0')+\text{Count}('H_{1\text{ or indecisive}}')}$.

For the decision probability of the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.37]$.

Asymmetrical Optional Stopping decisions for H_0 are extremely likely for small effect sizes and get more unlikely for increasing true means until it is extremely unlikely to decide for H_0. Deciding for H_0 with a small effect is not surprising, because the Asymmetrical Optional Stopping Procedure can be directly compared to the asymmetric Optional Stopping procedure in significance testing.

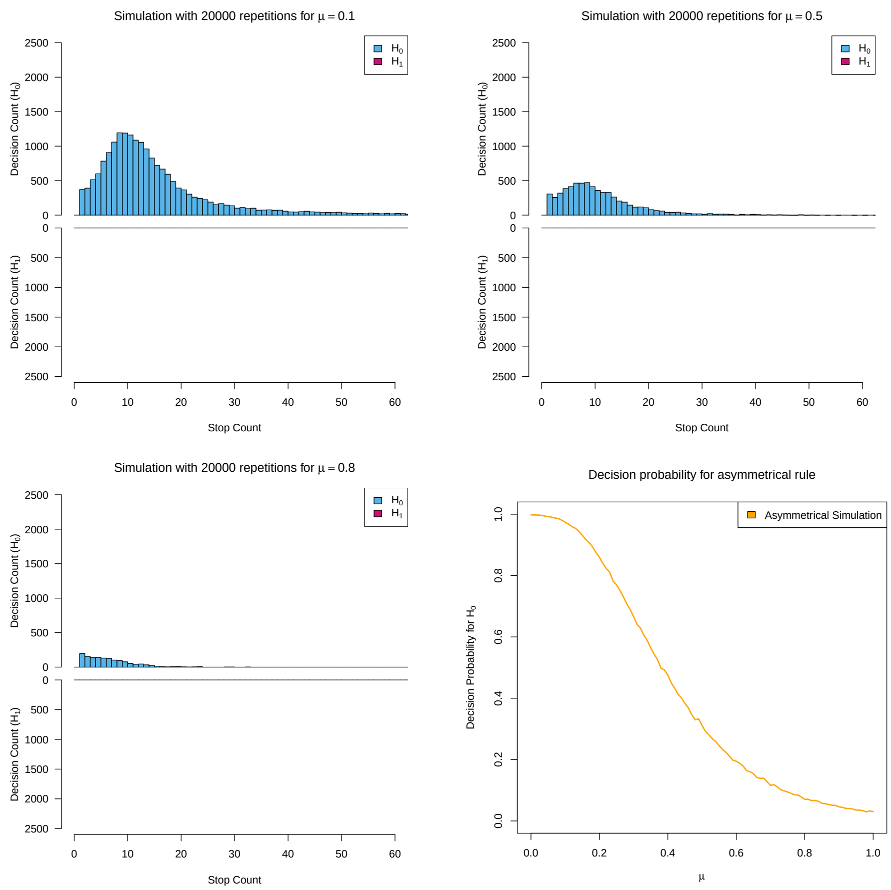


Figure 11: Decision Counts for H_0 and $H_{1 \text{ or indecisive}}$ decisions for each Stop Count. Decisions were made for all 20000 repetitions under the assumption of the Asymmetrical Optional Stopping Rule (See *Algorithm* 1). Note, decisions $H_{1 \text{ or indecisive}}$ were made after $n = 250$ data points.

On the other side for the symmetrical Optional Stopping simulation the results are rather surprising. Even though it is a symmetrical Optional Stopping procedure and therefore decides for H_0 or H_1 given the calculated Bayes Factors there is more than 80% probability to decide for the H_0 for small effects $\mu = [0, 0.2]$. For higher true means μ the decision probability for H_0 decreases.
The results for the symmetrical Optional Stopping simulation are shown in Figure 12.
Decision probability for H_0 was calculated here analogously to the asymmetric Optional Stopping simulation by taking the decision count of H_0 and divid-

ing it by the sum of the decision count of H_0 and H_1, such that $P('H_0') = \frac{\text{count}(H_0)}{\text{Count}(H_0)+\text{Count}(H_1)}$.

For the decision probability of the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.35]$.

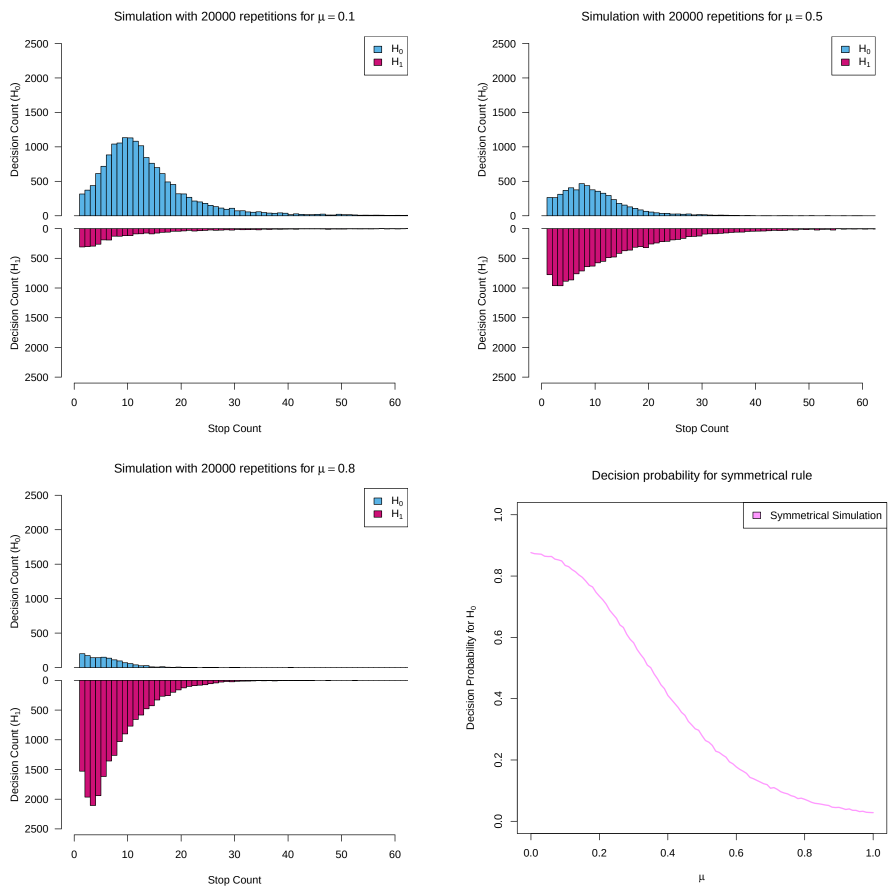


Figure 12: Decision Counts for H_0 or H_1 for each stop count. Decisions were made for all 20000 repetitions under the assumption of the symmetrical Optional Stopping procedure. On the right bottom a decision probability curve for H_0 over all $\mu = [0, 1]$ is shown.

Symmetrical and asymmetrical decision probabilities are compared to each other in Figure 13. The difference between asymmetrical and symmetrical is calculated as $P('H_0' \mid \mu, \text{diff}) = P('H_0' \mid \mu, \text{asymmetric}) - P('H_0' \mid \mu, \text{symmetric})$. The difference is the greatest for small true means μ, although it is noteworthy that for symmetrical Optional Stopping a lower probability for H_0 was expected and therefore the curves of the two Optional Stopping procedures fit together better than expected.

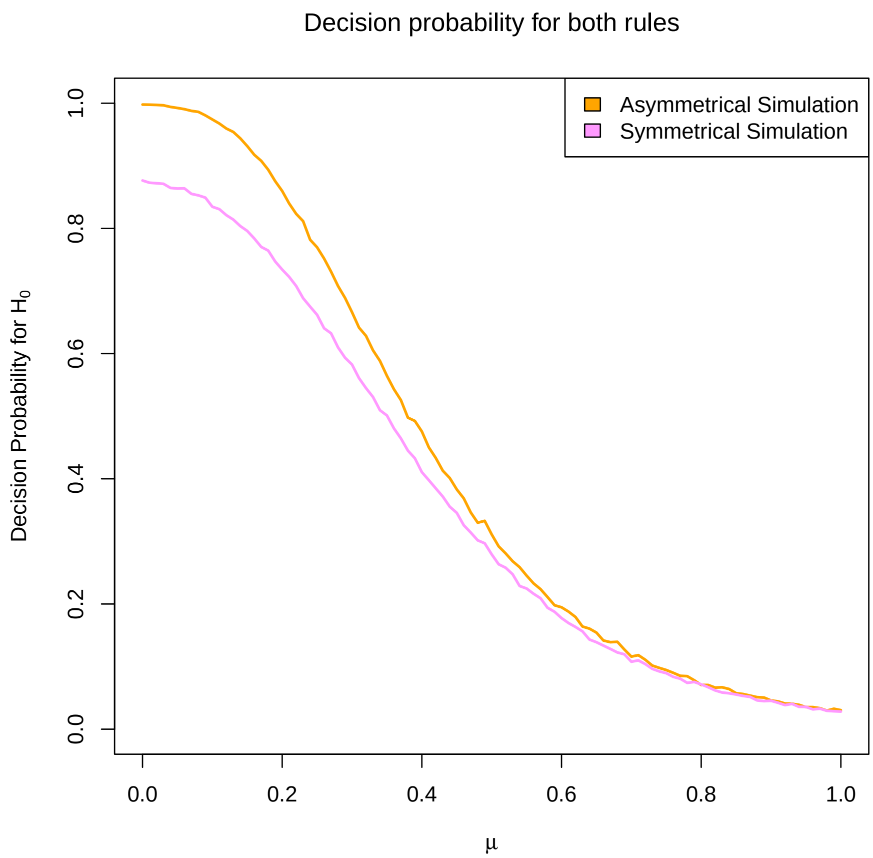


Figure 13: Asymmetrical Optional Stopping leads to a higher bias towards H_0 for smaller true effect sizes μ. However, the difference decreases with increasing μ.

3.2.2 Analytical Approximation for Normal Prior With Known Variance (Idealised Setting)

It would be interesting, if the Optional Stopping process with its decision probability for H_0 can be approximated. To define an approximation for Optional Stopping, another perspective on decision thresholds can be taken. Until now, in

Optional Stopping simulations decisions for H_0 or H_1 were made by specifying a Bayes Factor threshold BF_{crit} and sampling from a normal distribution with a given true mean μ and a fixed true standard deviation σ and fixed alternative prior width σ_1, such that $\sigma = 1 = \sigma_1$. One can change the perspective by using the inverse of the Bayes Factor formula $BF_{10}^{-1} = BF_{\text{crit}}$ specified in Definition 2 to obtain the decision threshold $\pm\bar{y}_{\text{crit}}$ for H_0.

These calculated values are critical mean thresholds $\bar{y}_{\text{crit}}$, so a decision can be made with the Optional Stopping procedure for a certain estimated mean $\bar{y}$ with a given Bayes Factor threshold BF_{crit} instead of a calculated Bayes Factor.

For the asymmetrical case it is assumed that $Y \sim \mathcal{N}(\mu, \sigma = 1)$, therefore we assume $\bar{y}(n) \sim \mathcal{N}(0, \frac{1}{\sqrt{n}})$. This visualization for fixed $\mu = 0.1$ can be seen in 14. The decision probability is given by the sum of the probabilities to stop at each step k and decide for H0:

$$P('H_0'|\text{stopping rule } r) = \sum_{k=1}^{\infty} P('H_0', N_{\text{stop}} = k)$$

Note, that the asymmetric stopping rule always decides for H_0 when it stops, therefore $P('H_0', N_{stop} = k) = P(N_{stop} = k)$. However $P('H_0', N_{stop} = k)$ is hard to derive since having not stopped before k non-trivially influences how the data is distributed at step k.
Therefore a look is taken at the following approximation, which neglects this dependency and assumes data at step k is distributed as it would have been for a fixed sample size:

$$\tilde{P}('H_0' \mid N_{stop} = k) := P('H_0' \mid N = k) \cdot \prod_{i=1}^{k-1} 1 - P('H_0' \mid N = i)$$

Note that it is still necessary to ensure that in all steps before k it was not stopped by using the same approximation. This can also be defined recursively:

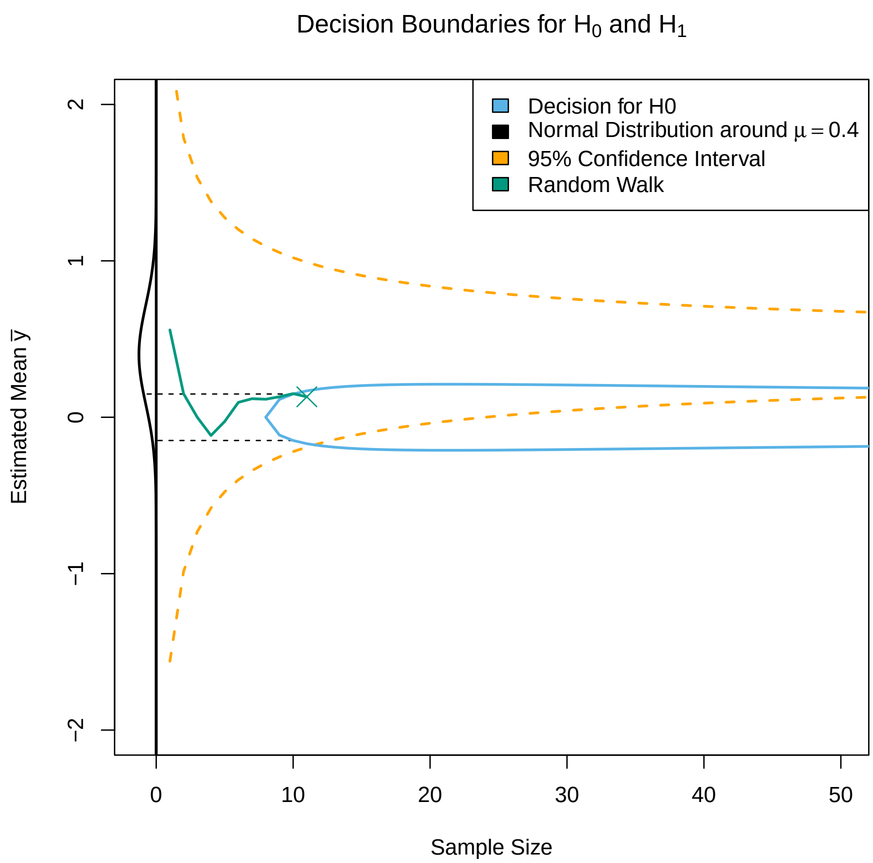


Figure 14: Asymmetrical Optional Stopping: An approximation on critical means $\pm\bar{y}_{crit_0}$ with the visualization of the deduction. The 95% Confidence Interval of the normal distribution around μ covers a broad area of the Decision Boundary of H_0. For increasing sample size n the interval narrows down.

$$\begin{aligned}
\tilde{P}('H_0' \mid N_{stop} = k) =& P('H_0' \mid N = k) \\
& \cdot \underbrace{\prod_{i=1}^{k-1} 1 - P('H_0' \mid N = i)}_{P(\text{`}H_1 \text{ or indecisive}' \mid N \leq k-1)} \\
P(\text{`}H_1 \text{ or indecisive}' \mid N \leq k-1) =& P(\text{`}H_1 \text{ or indecisive}' \mid N = k-1) \\
& \cdot \underbrace{\prod_{i=1}^{k-2} 1 - P('H_0' \mid N = i)}_{P(\text{`}H_1 \text{ or indecisive}' \mid N \leq k-2)}
\end{aligned}$$

Here $P('H_0', N = n)$ can be calculated via the cumulative density function Φ, because the distribution of $\bar{y}(n)$ as well as the critical values $\pm\bar{y}_{\text{crit}}(BF_{\text{crit}}, n, \sigma_1)$ are known:

$$\begin{aligned}
P(H_0 \mid N = n) &= \int_{-\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)}^{\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\partial x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)}^{\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\partial x \\
&= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)}^{\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \exp\left[-\frac{1}{2}\left(\sqrt{n}(x-\mu)\right)^2\right]\partial x \\
&= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)}^{\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \exp\left[-\frac{n}{2}(x-\mu)^2\right]\partial x \\
&= \frac{\sqrt{n}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \exp\left[-\frac{n}{2}(x-\mu)^2\right]\partial x\right. \\
&\quad \left.- \int_{-\infty}^{-\bar{y}_{\text{crit}}(BF_{\text{crit}},n,\sigma_1)} \exp\left[-\frac{n}{2}(x-\mu)^2\right]\partial x\right) \\
&= \Phi_{\mu,\sigma=\frac{1}{\sqrt{n}}}\left(\bar{y}_{\text{crit}}(BF_{\text{crit}}, n, \sigma_1)\right) \\
&\quad - \Phi_{\mu,\sigma=\frac{1}{\sqrt{n}}}\left(-\bar{y}_{\text{crit}}(BF_{\text{crit}}, n, \sigma_1)\right)
\end{aligned}$$

This approximation for $P('H_0', N_{stop} = k)$ also leads to the following approximation for the decision probability:

$$
\begin{aligned}
\tilde{P}('H_0' \mid \text{stopping rule } r) &= \sum_{k=1}^{\infty} \tilde{P}('H_0', N_{stop} = k) \\
&= \sum_{k=1}^{\infty} \left[P('H_0' \mid N = k) \cdot \prod_{i=1}^{k-1} [1 - P('H_0' \mid N = i)] \right] \\
&= \sum_{k=1}^{\infty} \Bigg[\Phi_{\mu,\sigma=\frac{1}{\sqrt{k}}} \left(\bar{y}_{\text{crit}}(BF_{\text{crit}}, k, \sigma_1)\right) \\
&\quad - \Phi_{\mu,\sigma=\frac{1}{\sqrt{k}}} \left(-\bar{y}_{\text{crit}}(BF_{\text{crit}}, k, \sigma_1)\right) \\
&\quad \cdot \prod_{i=1}^{k-1} \Big[1 - \big(\Phi_{\mu,\sigma=\frac{1}{\sqrt{i}}} \left(\bar{y}_{\text{crit}}(BF_{\text{crit}}, i, \sigma_1)\right) \\
&\quad - \Phi_{\mu,\sigma=\frac{1}{\sqrt{i}}} \left(-\bar{y}_{\text{crit}}(BF_{\text{crit}}, i, \sigma_1)\right) \big) \Big] \Bigg]
\end{aligned}
$$

The results for the approximation in comparison to the Optional Stopping simulation are shown in Figure 15.

The approximation for the asymmetrical Optional Stopping simulation is not great as it does not look like a lower or upper bound of the simulation. Instead it is at $P('H_0') = 1$ for $\mu = [0, 0.5]$ and decreases only afterwards with a big distance to the simulation. For the decision probability of the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.64]$. Therefore it is not a very good approximation and will not be considered.

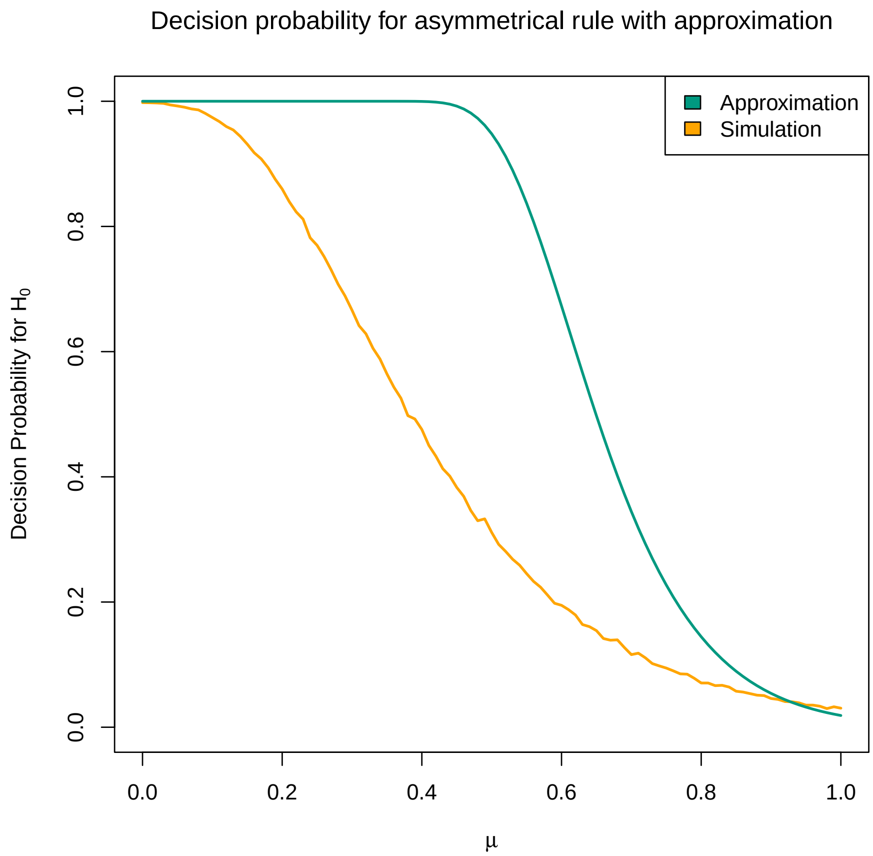


Figure 15: The probability $P('H_0')$ to decide for H_0 via critical mean thresholds $\pm\bar{y}_{\text{crit}}$ for all true means $\mu = [0, 1]$ with step size $\delta = 0.01$ in the Optional Stopping simulation and its approximation.

The deduction of the approximation for the symmetrical Optional Stopping simulation is analogous to the asymmetrical case in addition to the density calculation for the alternative hypothesis H_1. For more details, see Appendix A.2.

The results for the approximation in comparison to the Optional Stopping simulation in the symmetrical case are shown in Figure 16. The approximation fits the shape of the simulation. It predicts at all times smaller probabilities for a H_0 decision than the simulation decides for. For the decision probability of the approximation with respect to the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.26]$.

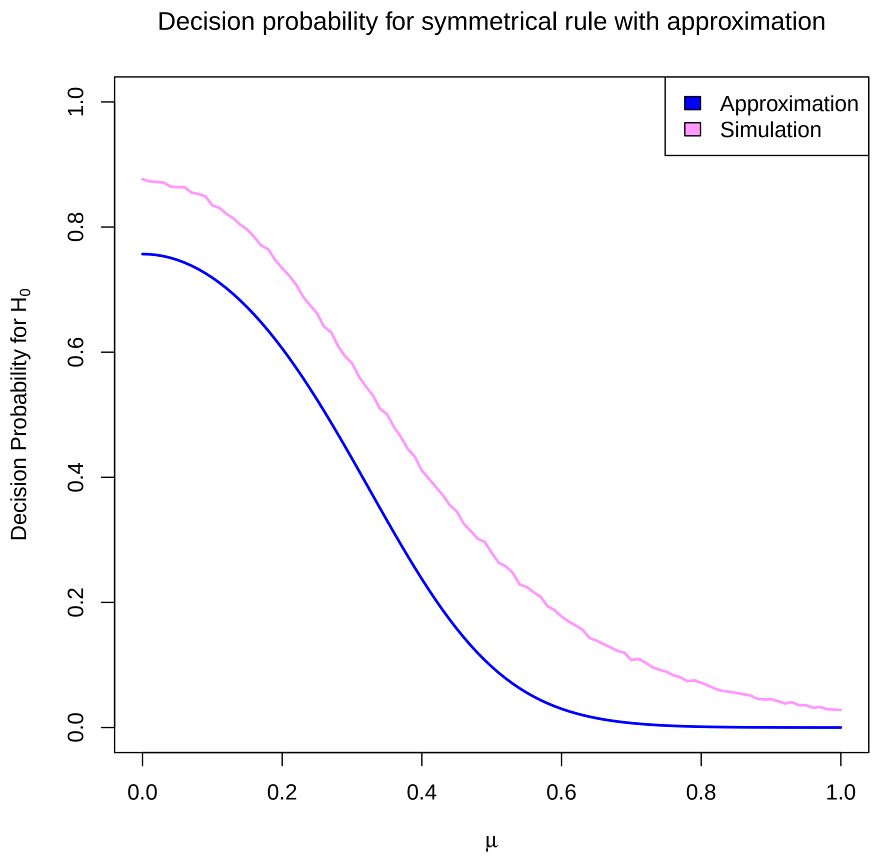
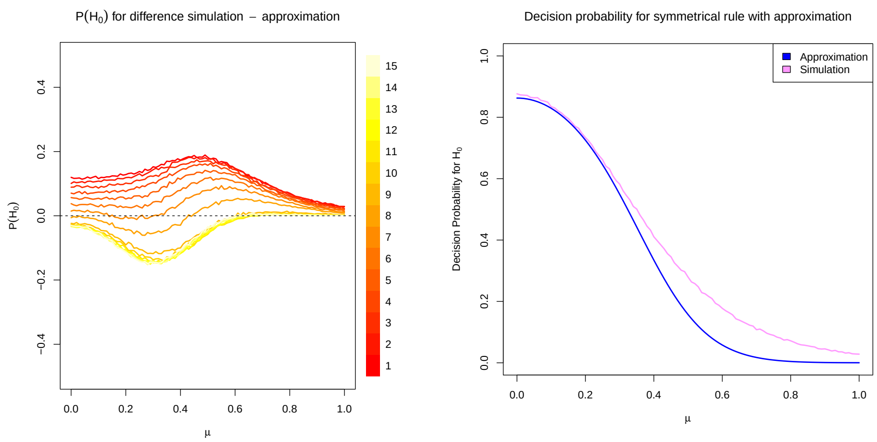


Figure 16: The results for critical means $\bar{y}_{\text{crit}}$ with the probability $P('H_0')$ to decide for null hypothesis H_0 for all $\mu = [0, 1]$ for both, the simulation and approximation.

There is a certain trade-off in Figure 16 between the densities of the decision boundaries for H_0 and H_1, which leads to an overestimation of the decisions for H_1 in the beginning of the Optional Stopping process. This is the case, because until the density at the sample size k of the critical mean boundaries for H_0 can be estimated the probability for a decision for H_0 is $P('H_0' \mid n < k) = 0$
To improve the approximation a search for the optimal starting count of gathered data points for the Optional Stopping process n_{start} is conducted. The search space is set in the interval $n_{\text{start}} = [1, 15]$ — as the decision boundary of

H_0 is starting at $n_{\text{start}} = 8$ — because it is unlikely that the density of the decision boundary of H_0 is less than the empirical simulation and instead the density of the decision boundary of H_0 is overestimated. The Optional Stopping simulation and the approximation were again conducted with these changed starting values and compared with the corresponding starting values n_{start}. The results can be seen in Figure 17a.

The optimal n_{start} has been chosen by the smallest positive difference for $P('H_0' \mid \mu, \text{symmetric simulation})$ - $P('H_0' \mid \mu, \text{symmetric approximation})$ for each fixed n_{start}. The best fit for the approximation is $n_{\text{start}} = 6$ and is visualized in Figure 17b. For the decision probability of the approximation with respect to the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.32]$.



(a) The error of the approximation towards the Optional Stopping simulation for different gathered data starting points for the Optional Stopping simulation and approximation n_{start}.

(b) The decision probability for H_0 given μ with $n_{\text{start}} = 6$ for the approximation compared to the simulation - both in the symmetrical case.

3.2.3 Cauchy-Prior With Unknown Variance (Realistic Setting)

For the realistic setting only the symmetrical Optional Stopping procedure was simulated. The main difference is the unknown variance in the cauchy prior. The results of the Optional Stopping simulation are visualised in Figure 18.

It is surprising that the symmetrical Optional Stopping procedure yield very similar results in the realistic setting compared to the idealised setting. Again, for small effect sizes $\mu = [0, 0.2]$ almost 70% of the decisions are made in favor of the null hypothesis H_0. For higher effects this behaviour again decreases.

For the decision probability of the null hypothesis $P('H_0') > 0.5$ the true mean interval is $\mu = [0, 0.32]$.
Note, that decisions for H_0 start at sample size $n = 8$. This seems to be likely due to the chosen scale r, because in the simulations for lower r-scales the needed sample size to decide for H_0 increases. For $r = \frac{0.5}{\sqrt{2}}$ the required sample size is $n = 32$. For $r = \frac{2}{\sqrt{2}}$ the required sample size is $n = 2$.

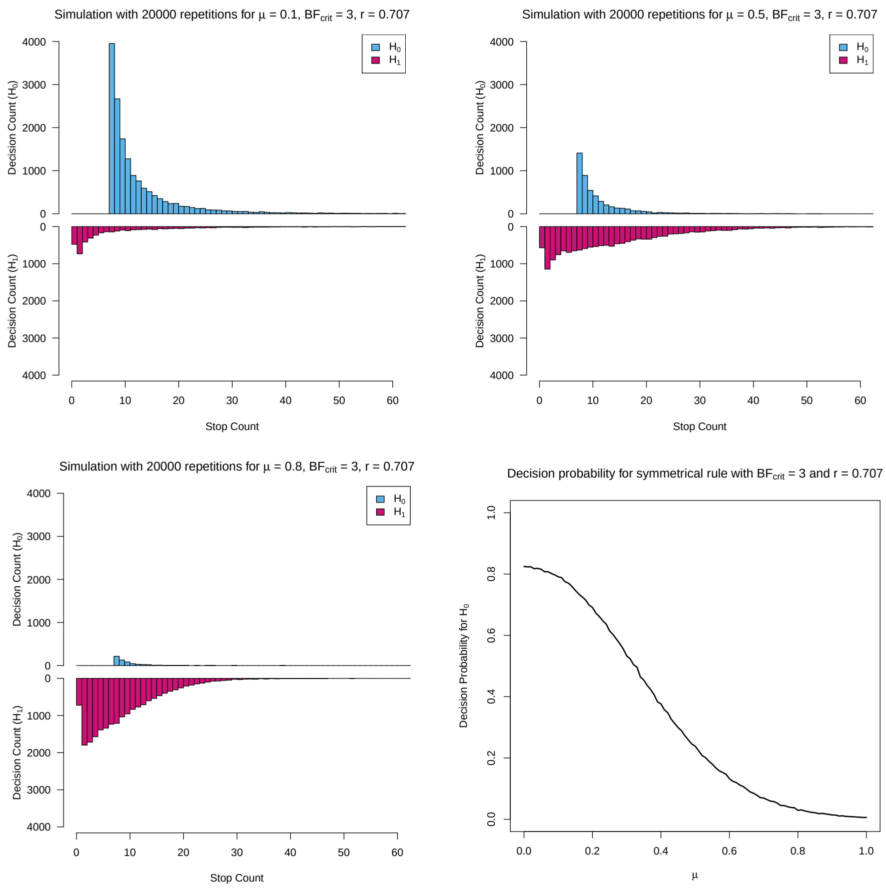


Figure 18: Decision Counts for H_0 or H_1 for each stop count. Decisions were made for all 20000 repetitions under the assumption of the symmetrical Optional Stopping procedure. On the right bottom a decision probability curve for H_0 over all $\mu = [0, 1]$ is shown.

A summary of the different combinations of scale r, true effect sizes μ and Bayes Factor decision thresholds BF_{crit} can be seen in Figure 19. Across the three plots for increasing BF_{crit} the probability to decide for the null hypothesis H_0 decreases. Also, with an increase in scale r the decision probability for H_0 increases as well.

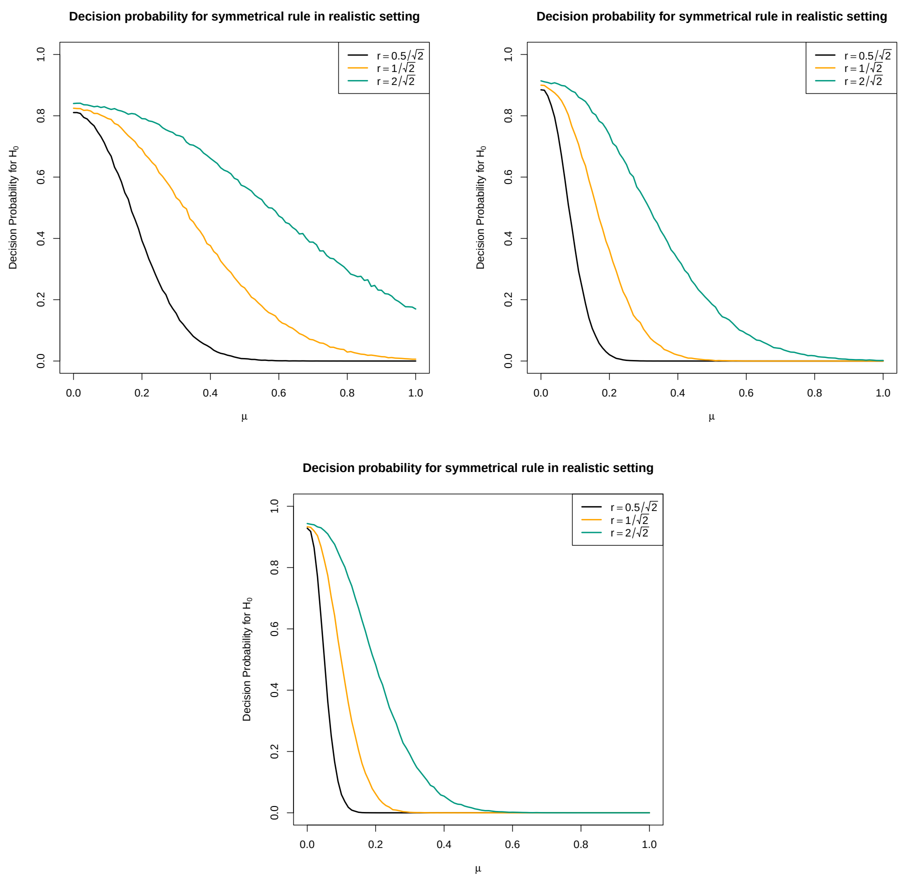


Figure 19: Symmetrical Simulation with Cauchy Prior for different r and BF_{crit} in a true mean interval of $\mu = [0, 1]$ with a step size of $\delta = 0.01$. Left top: $BF_{\text{crit}} = 3$, right top: $BF_{\text{crit}} = 6$ and bottom: $BF_{\text{crit}} = 10$.

3.3 Application Example: How to hack Bayes Factors? Change the parameters!

Assume the idealised setting, but one can assume a true effect of $\mu = 0.1$ and a true variance of $\sigma^2 = 1$. The goal is to use the Catch Up Effect properties in this setting for an Optional Stopping simulation to

1. Maximize the probability of the null hypothesis $P('H_0')$
2. Maximize the probability of the alternative hypothesis $P('H_1')$

There are several parameters that one can adjust: variance σ^2, alternative prior width σ_1, the Bayes Factor decision threshold BF_{crit} and Optional Stopping sample size minimum n_{start} and Optional Stopping sample size maximum n_{end}.

Maximize the decision probability of the null hypothesis $P('H_0')$

- Choose a "noisy" $\sigma^2 > 1$. This is possible if noise is deliberately introduced e.g. in experimental setups. Assume that noise is introduced to get to a variance of $\sigma^2 = 4$.
- Choose σ_1 for $\mu < \sigma_1$. Because μ is estimated or known it is easy to choose a σ_1 that is greater than μ. However, if σ_1 increases, the probability to decide for H_0 increases as well, so it is wise to choose a higher σ_1. It is also known in literature, that a chosen alternative prior width, that is too vague, is misspecified for sure, so σ_1 is chosen at the higher bound of recommendation, $\sigma_1 = 3$ (see Tendeiro and Kiers (2019), p. 779f.).
- Choose a low Bayes Factor threshold BF_{crit} to increase the chance the Catch Up Effect occurs. The lowest reasonable Bayes Factor decision threshold to choose is typically $BF_{\text{crit}} = 3$.
- Calculate n_{start} and n_{end} based on $1/BF_{\text{crit}} = 1/3$. For this the proposed algorithm to find intersection points between Bayes Factor thresholds and BF_{10} is used (*Algorithm* 3). Therefore $n_{\text{start}} = \lceil 3.56 \rceil = 4$ and $n_{\text{end}} = \lfloor 2589.72 \rfloor = 2590$. It is known, that it may be likely between the interval $n = [4, 2590]$ to decide for H_0. Therefore it is wise to decide for $n_{\text{start}} = 4$ as start sample size to start Symmetrical Optional Stopping with and $n_{\text{end}} = 2590$ as stopping sample size — if no decision has been made until then.

The resulting probability for the null hypothesis is $P('H_0') = 0.821$. The results can be seen in Figure 20.

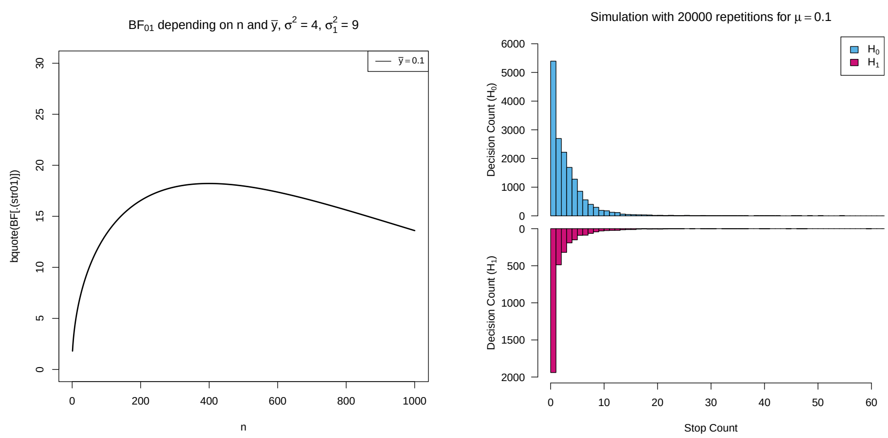


Figure 20: $\sigma_1^2 = 3$ and $\sigma = 2$. On the left is the BF_{01} function. On the right is a Symmetrical Optional Stopping simulation with $n_{\text{start}} = 4$ and a stopping condition for $n_{\text{end}} = 2590$. Note that in the simulation Stop Count $n = 0$ equals $n_{\text{start}} = 4$.

Maximize the decision probability of the alternative hypothesis $P('H_1')$

- Minimize σ^2 as much as possible. Lets say it is possible to minimize noise as much as possible, so the resulting variance is $\sigma^2 = 1$.
- Choose σ_1 close to μ, such that σ_1 slightly greater than μ. Because μ is known set $\sigma_1 = 0.15$.
- Choose a high Bayes Factor threshold BF_{crit} to decrease the chance the Catch Up Effect occurs. The Bayes Factor threshold, however, is set to $BF_{\text{crit}} = 3$.
- Calculate n_{start} and n_{end} based on $1/BF_{\text{crit}} = 1/6$. For this the proposed algorithm to find intersection points between Bayes Factor thresholds and BF_{10} is used (*Algorithm* 3). Therefore $n_{\text{start}} = \{\}$ and $n_{\text{end}} = \{\}$ and no start sample or stopping sample size is set.

Probability for the null hypothesis is $P('H_0' \mid BF_{\text{crit}} = 3) = 0.916$. For a given $BF_{\text{crit}} = 6$ it is even higher $P('H_0' \mid BF_{\text{crit}} = 6) = 0.999$. The results can be seen in Figure 21.

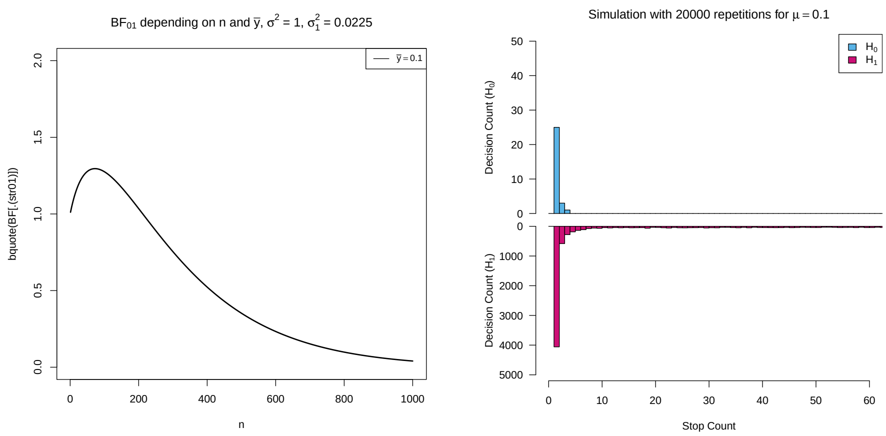


Figure 21: $\sigma_1 = 0.15$ and $\sigma = 1$. On the left is the BF_{01} function. On the right side is the histogram for a symmetrical Optional Stopping simulation with $n_{\text{start}} = 1$ and no n_{end}.

4 Discussion

The aim of this exploratory bachelor thesis is to determine if Bayesian Optional Stopping is really unproblematic as it is claimed by Rouder (2014). This was done by looking at the Bayes Factors in an idealised setting (point vs. normal prior, see Tendeiro and Kiers (2019)) and in a more realistic setting (point vs. Cauchy prior). For the former the Bayes Factor function BF_{10} and its properties — especially for the non monotonic behaviour of BF_{10}, the Catch Up Effect — was investigated and Optional Stopping Simulations for the asymmetric and the symmetric Optional Stopping procedure conducted. The Catch Up Effect in the realistic setting was investigated by conducting an Optional Stopping simulation for the symmetric Optional Stopping procedure.
The Catch Up Effect analyis of the idealised setting for BF_{10} shows, that there always exist exactly one minimum (if $\bar{y} \neq 0$) , whose value is not influenced by σ^2, but whose location scales with it. If for BF_{10} the amount of the mean $|\bar{y}|$ increases then BF_{10} increases. If for BF_{10} the alternative prior width σ_1 with $\sigma_1 > \bar{y}$ increases, then BF_{10} decreases and converges for very large σ_1 to 0. If σ_1 with $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$ decreases, than BF_{10} decreases as well, although for very small σ_1 it converges to 1. For a given Bayes Factor decision threshold it is also possible to calculate a start sample size n_{start} and a sample size n_{end} for Optional Stopping to influence the decision probabilities.
It is not surprising that the minimum location scales with known variance σ^2. The greater the noise and true variance are, the greater the sample size has to be to make a decision. Similar to this, it is also not surprising, that σ^2 is independent to the minimum value. The independence of the minimum value of the effect is not surprising either. It is comprehensive, that Bayes Factors and the minimum of the Catch Up Effect do not depend on the known variance of the data, but only on the observed mean, prior width variance and sample size. The increase of BF_{10} with the increased amount of observed mean $\bar{y}$ is a direct consequence of the central limit theorem: while the sample size increases towards infinity, the Bayes Factors converges towards the alternative decision. This is a crucial property of hypothesis tests for researchers to make reliable decisions for the alternative hypothesis if a true effect is existent.
The analysis shows the alternative prior width σ has a more unintuitive interpretation: If $\sigma_1 < \sqrt{\bar{y}^2 - \sigma^2}$ then BF_{10} decreases as the alternative prior width decreases. However, in practice this is in itself a very small alternative prior width. As this width decreases towards 0, the Bayes Factor converges towards

1. Therefore, if such a small alternative prior width is chosen, the density of the normal prior converges to the density of the point prior. Bayes Factors can not differentiate between the null or alternative hypothesis.
If $\sigma_1 > \bar{y}$ then BF_{10} decreases as σ_1 increases. This behaviour was also reported by Tendeiro and Kiers (2019). If a higher standard prior width is specified, it is more likely to decide for the null hypothesis. However, if someone specifies the standard prior width smaller than the Catch Up Effect decreases and a more likely decision for the alternative hypothesis is made. This means that one can misspecify the prior so small, that a decision towards the alternative hypothesis can be influenced.
Start sample size n_{start} and end sample size n_{end} for a given Bayes Factor threshold BF_{crit} can be calculated by Newtons Method. Intuitively the most likely start sample size n_{start} and end sample size n_{end} influences the decision for the null hypothesis with the corresponding decision threshold BF_{crit}.
The Optional Stopping simulation in the idealised setting uncovered, that there is a high bias for decision towards H_0 for small to medium sized effects ($P('H_0') \geq 0.8$ for $\mu = [0, 0.2]$). However, this is also the case for the Optional Stopping simulation conducted for the realistic setting, although with a lesser decision probability for H_0 ($P('H_0') \geq 0.7$ for $\mu = [0, 0.2]$).
There are many parameters to adjust for increasing the decision probability for the null hypothesis - the Optional Stopping simulations only show the adjustments for a true mean μ with a fixed alternative prior width $\sigma_1 = 1$. The results clearly show, that there is a big overestimation for the null hypothesis - especially for small effect sizes. This overestimation was reported by Rouder et al. (2009) "[if σ_1] is set unrealistically high", but the setting of $\sigma_1 = 1$ is seen as "reasonable"(Rouder et al., 2009, p.230). The Catch Up Effect of the null hypothesis seemingly was not known and underestimated by researchers, which lead to an overconfidence in default priors. It is surprising that not only the trivial, asymmetrical Optional Stopping procedure shows a clear bias towards the null hypothesis, but even the symmetrical Optional Stopping with a Cauchy prior as alternative prior shows a substantial bias for the null.
Just as Tendeiro and Kiers (2019) pointed out, it might also be the case, that the interpretation of Bayes Factors without contextualising the meaning of their value or discarding the posterior is problematic for making decisions in Optional Stopping. Lets say that there is a Bayes Factor of value 3. The common interpretation is that this means it is just three times more likely than the compared model. But what exactly does this mean? Especially in model specifications

with an alternative model where default priors (e.g. Cauchy priors) are used in comparison to the null model (point prior) it is not entirely clear how the inherent belief for a hypothesis inherently correlates to the chosen models (Tendeiro & Kiers, 2019, p. 782f.). The value of a Bayes Factor is highly dependent on the chosen within-prior models (Tendeiro & Kiers, 2019, p.778f.).

A Bayesian t-test, that would be able to make reliable decisions in an Optional Setup framework would be a desirable property. It would be cost-efficient, because one only needs a sample size that is just big enough to make a decision. However, as pointed out, the bias in form of the Catch Up Effect between the effect size and decisions made for the null hypothesis is more impactful than commonly thought. Researchers need to carefully examine for their assumed, to be measured, effects. It can then be problematic, if they want to use Optional Stopping with the Bayesian t-test for their effect size. Especially, if they wrongly estimate the effect size and the corresponding relevant parameters it is highly likely to make decisions for the null hypotheses even though an effect exists.

The application example shows, by using the properties of BF_{10} in the idealised setting one can change these parameters to maximize the likelihood for a decision towards the null hypothesis H_0 or towards H_1.

Therefore there are many ways to influence Bayes Factors with default priors. It is easy to misspecify parameters like $\sigma^2, \sigma_1, BF_{crit}$, etc. in practice, so Bayes Factors are easy to hack. If the underlying inherent belief is not specified and the selection of the within-priors is not disclosed, there is no way to distinguish a reasonable Bayes Factor from a hacked Bayes Factor. The "objectivity" of a default prior seems to not align with the Bayesian statistical philosophy in which one updates their inherent beliefs according to the observed data. If a researcher does not have a subjective, justified reason to believe in the prior as a correct model for their hypothesis it is not reasonable to choose. The consideration for default priors just because of the "appearance of objectivity instead of true objectivity" (Tendeiro & Kiers, 2019, p.781) lead to problematic interpretations of the resulting Bayes Factors.

Furthermore, a reliable interpretation of the choice of the Cauchy scale r and the alternative prior width σ_1 is needed, if default priors are used for the representation of a justified beliefs. Researchers should get an intuition for which different r or σ_1 are justified and when it is even helpful to rely on a default prior.

As de Heide and Grünwald (2021) already concluded, Bayesian Optional Stopping can be used for subjective priors that justify the beliefs of the researcher about the data. Default priors should only be used if there is a reasonable belief. They are unfit to detect small true effects in the current implementation of the Bayesian t-test. Additionally it is advised to take a sufficiently large start sample size and to not stop at a specific end sample size. However, even if one does account for this and misspecifies the prior, then small true effects still require a very large sample size, not accounting for noise and variance.

The main focus of this thesis was on the idealised setting, which incorporated the normal prior as alternative prior. Even though an Optional Stopping simulation on the Cauchy prior was already conducted it is still unclear if properties of the Catch Up Effect observed in the point vs. normal prior Bayes Factor occurs in the point vs. Cauchy prior Bayes Factor and how they relate if the variance is unknown. Also the relation between the Catch Up Effect and the Optional Stopping simulation should be further investigated. The different influences of the Catch Up Effect and Optional Stopping are clear, but how they relate to each other and which parameters are more dominant over each other is still needs to be answered.

Default priors should not be used without a justified belief. The parameter choice and their reasons should be disclosed. Optional Stopping with the Catch Up Effect shows that Bayes Factors can be hacked for default priors to influence a decision towards either hypothesis. The promise of NHBT that Optional Stopping is possible requires qualification: There is no free lunch and, when done incorrectly, incorrect interpretations can come from using Optional Stopping.

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A Appendix

A.1 Generalizing $\bar{y}_{\text{crit}}$ for $BF_{10} = BF_{\text{crit}}$

$$
\begin{aligned}
BF_{10} &= BF_{\text{crit}} \\
\exp\left[\frac{n^2\sigma_1^2\bar{y}_{\text{crit}}^2}{2(1+n\sigma_1^2)}\right] &= BF_{\text{crit}} \cdot \sqrt{1+n\sigma_1^2} \\
\frac{n^2\sigma_1^2\bar{y}_{\text{crit}}^2}{2(1+n\sigma_1^2)} &= \ln\left(BF_{\text{crit}}\sqrt{1+n\sigma_1^2}\right) \\
n^2\sigma_1^2\bar{y}_{\text{crit}}^2 &= 2(1+n\sigma_1^2)\ln\left(BF_{\text{crit}}\sqrt{1+n\sigma_1^2}\right) \\
\bar{y}_{\text{crit}}^2 &= \frac{2(1+n\sigma_1^2)\ln\left(BF_{\text{crit}}\sqrt{1+n\sigma_1^2}\right)}{n^2\sigma_1^2} \\
\bar{y}_{\text{crit}} &= \pm\sqrt{\frac{2(1+n\sigma_1^2)\ln\left(BF_{\text{crit}}\sqrt{1+n\sigma_1^2}\right)}{n^2\sigma_1^2}} \\
\bar{y}_{\text{crit}} &= \pm\frac{\sqrt{2(1+n\sigma_1^2)}}{n\sigma_1}\left(\ln\left(BF_{\text{crit}}\sqrt{1+n\sigma_1^2}\right)\right)^{1/2}
\end{aligned}
$$

After substituting $BF_{\text{crit}} = 1$ one can retrieve the corresponding formula of Tendeiro and Kiers, 2019:

$$
\bar{y}_{\text{crit}} = \pm\frac{\sqrt{2(1+n\sigma_1^2)}}{n\sigma_1}\left(\ln\left(\sqrt{1+n\sigma_1^2}\right)\right)^{1/2} \tag{11}
$$

A.2 Symmetrical Optional Stopping Approximation

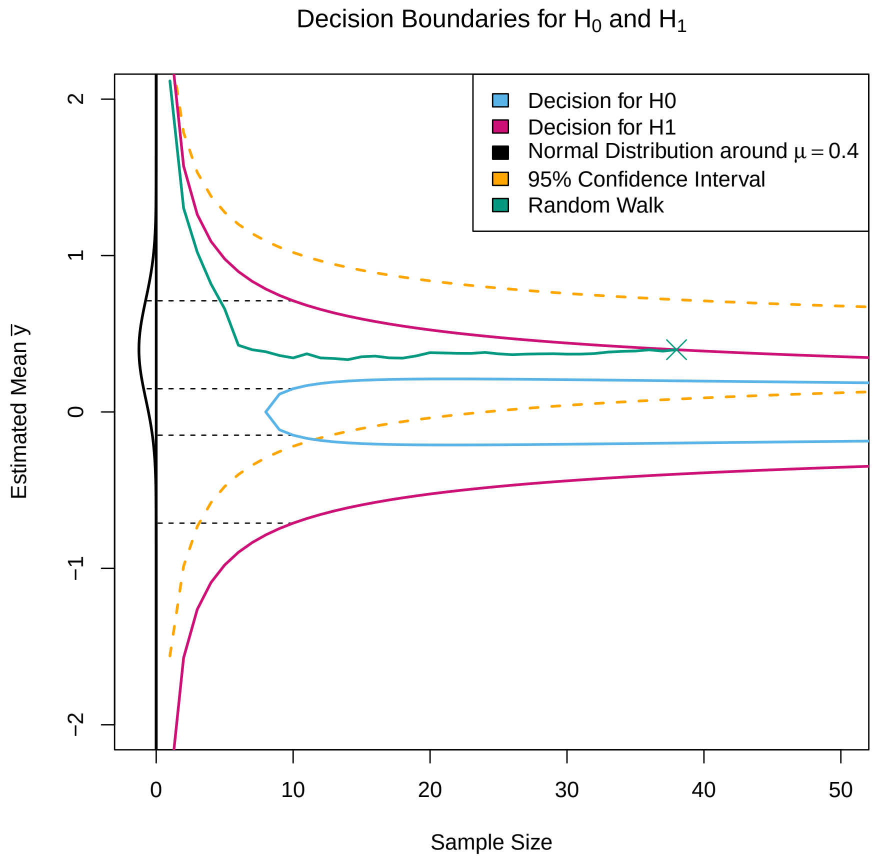

Figure A1: Symmetrical Optional Stopping: An approximation on critical means $\bar{y}_{\text{crit}}$ with the visualization of the deduction. The 95% Confidence Interval of the normal distribution around μ covers some area of the Decision Boundaries of H_0 and H_1. For increasing n the interval narrows down.

Assume data $y \sim \mathcal{N}(\mu, \sigma = 1)$, therefore it is assumed $\bar{y}(n) \sim \mathcal{N}(0, \frac{1}{\sqrt{n}})$. The decision probability is given by the sum of the probabilities to stop at each step k and decide for H_0 or H_1:

$$P('H_0 \text{ or } H_1'|\text{stopping rule } r) = \sum_{k=1}^{\infty} P('H_0 \text{ or } H_1', N_{\text{stop}} = k)$$

Now, for the symmetric stopping rule, for each stop a decision is made either for H_0 or H_1. Therefore

$$\begin{aligned} P('H_0 \text{ or } H_1', N_{\text{stop}} = k) &= P('H_0', N_{\text{stop}} = k) + P('H_1', N_{\text{stop}}) \\ &= P(N_{stop} = k) \end{aligned}$$

Looking at the following approximation as $P('H_0 \text{ or } H_1', N_{\text{stop}} = k)$ is hard to derive since having not stopped before k non-trivially influences how the data is distributed at step k, that neglects this dependency and assumes data at step k is distributed as it would have been for a fixed sample size.

$$\tilde{P}('H_0' \mid N_{stop} = k) := P('H_0' \mid N = k) \cdot \prod_{i=1}^{k-1} 1 - P('H_0 \text{ or } H_1' \mid N = i)$$

$$\tilde{P}('H_1' \mid N_{stop} = k) := P('H_1' \mid N = k) \cdot \prod_{i=1}^{k-1} 1 - P('H_0 \text{ or } H_1' \mid N = i)$$

Similar to the symmetric case we can calculate $P('H_0', N = n)$ and $P('H_1', N = n)$ via the cumulative densitiy function Φ, because we know the distribution of $\tilde{y}(n)$ as well as the critical values $\pm y_{\text{crit}}(BF_{crit_1}, n, \sigma_1)$ and $\pm y_{\text{crit}}(BF_{crit_2}, n, \sigma_1)$:

$$\begin{aligned} P('H_0' \mid N = n) = \Phi_{\mu, \sigma = \frac{1}{\sqrt{n}}} &(\bar{y}_{\text{crit}}(BF_{crit_1}, n, \sigma_1)) \\ - \Phi_{\mu, \sigma = \frac{1}{\sqrt{n}}} &(-\bar{y}_{\text{crit}}(BF_{crit_1}, n, \sigma_1)) \end{aligned}$$

$$\begin{aligned} P('H_1' \mid N = n) = \Phi_{\mu, \sigma = \frac{1}{\sqrt{n}}} &(\bar{y}_{\text{crit}}(BF_{crit_2}, n, \sigma_1)) \\ - \Phi_{\mu, \sigma = \frac{1}{\sqrt{n}}} &(-\bar{y}_{\text{crit}}(BF_{crit_2}, n, \sigma_1)) \end{aligned}$$

These approximations for $P('H_0', N_{stop} = k)$ lead to the following approximation for the decision probability:

$$
\begin{aligned}
\tilde{P}('H_0' \mid \text{stopping rule } r) &= \sum_{k=1}^{\infty} \tilde{P}('H_0', N_{stop} = k) \\
&= \sum_{k=1}^{\infty} \Bigg[P('H_0' \mid N = k) \\
&\quad \cdot \prod_{i=1}^{k-1} [1 - P('H_0' \text{ or } 'H_1' \mid N = i)] \Bigg] \\
&= \sum_{k=1}^{\infty} \Bigg[P('H_0' \mid N = k) \\
&\quad \cdot \prod_{i=1}^{k-1} [1 - P('H_0' \mid N = i) - P('H_1' \mid N = i)] \Bigg] \\
&= \sum_{k=1}^{\infty} \Bigg[\Phi_{\mu, \sigma = \frac{1}{\sqrt{k}}} \left(\bar{y}_{\text{crit}}(BF_{crit_1}, k, \sigma_1) \right) \\
&\quad - \Phi_{\mu, \sigma = \frac{1}{\sqrt{k}}} \left(-\bar{y}_{\text{crit}}(BF_{crit_2}, k, \sigma_1) \right) \\
&\quad \cdot \prod_{i=1}^{k-1} \Big[1 - \big(\Phi_{\mu, \sigma = \frac{1}{\sqrt{i}}} \left(\bar{y}_{\text{crit}}(BF_{crit_1}, i, \sigma_1) \right) \\
&\quad - \Phi_{\mu, \sigma = \frac{1}{\sqrt{i}}} \left(-\bar{y}_{\text{crit}}(BF_{crit_1}, i, \sigma_1) \right) \big) \\
&\quad - \big(1 - \Phi_{\mu, \sigma = \frac{1}{\sqrt{i}}} \left(\bar{y}_{\text{crit}}(BF_{crit_2}, i, \sigma_1) \right) \\
&\quad + \Phi_{\mu, \sigma = \frac{1}{\sqrt{i}}} \left(-\bar{y}_{\text{crit}}(BF_{crit_2}, i, \sigma_1) \right) \big) \Big] \Bigg]
\end{aligned}
$$

List of Algorithms

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B Declaration of Authorship

Hiermit erkläre ich, dass ich diese schriftliche Abschlussarbeit selbständig verfasst habe, keine anderen als die angegebenen Hilfsmittel und Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe. Diese Bachelorarbeit wurde in gleicher oder ähnlicher Form in keinem anderen Studiengang als Prüfungsleistung vorgelegt.

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